What is the shape of a spinning black hole?

From geometry to gravitational-wave observables

Alexandre Le Tiec

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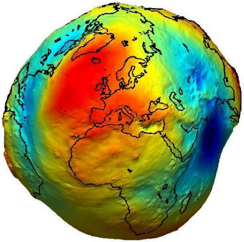
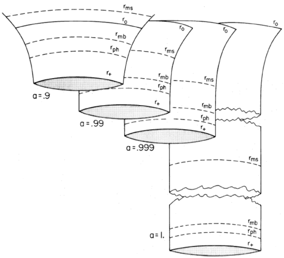
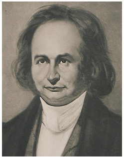


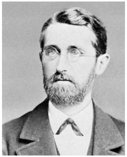


Some historical context

Colin Maclaurin
(1698-1746)

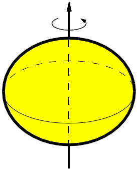


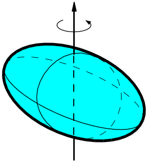
Carl G. J. Jacobi
(1804-1851)

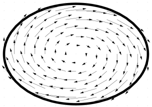


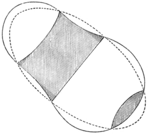
Richard Dedekind
(1831-1916)

Henri Poincaré
(1854-1912)

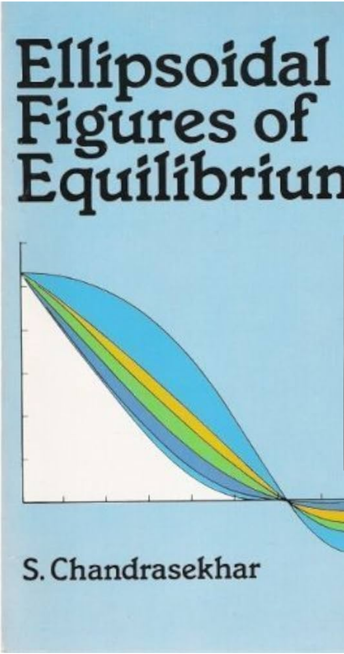


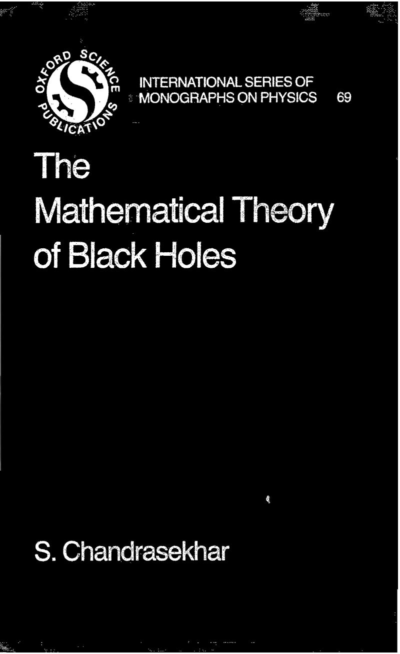






But also: Newton (1689), Huygens (1690), Cassini (1701), Maupertuis (1732), Clairaut (1733), Euler (1740), D'Alembert (1756), Lagrange (1759), Laplace (1772), Legendre (1784), Monge (1787), Poisson (1811), Gauss (1813), Cauchy (1815), Dirichlet (1857), Riemann (1860), Darwin (1906), Jeans (1917), Cartan (1924), ...


Ellipsoidal Figures of Equilibrium
S. Chandrasekhar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The Mathematical Theory of Black Holes
S. Chandrasekhar

Three complementary notions of "shape"

1. **Asymptotic shape** (field multipoles)
   - Geometry at spatial infinity
   - Geroch–Hansen multipoles

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3. **Dynamical shape** (tidal deformations)
   - Response to external fields
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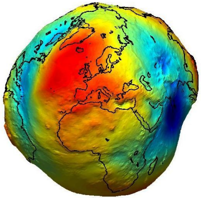
Goal: bridge geometric notions of "shape" with **GW observables**

1. Field multipoles
2. Horizon geometry
3. Tidal deformations
4. Gravitational-wave imprint

Field multipoles in Newtonian gravity

- Gravitational field:

$$U(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3\boldsymbol{x}'$$



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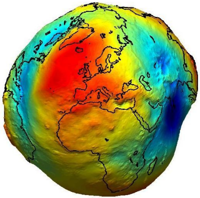

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- Multipole expansion:

$$U(r, \theta, \phi) = \sum_{l \geqslant 0} \sum_{|m| \leqslant l} \frac{4\pi}{2l+1} \frac{M_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi)$$

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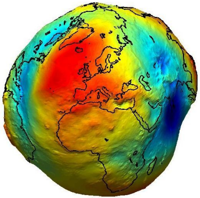

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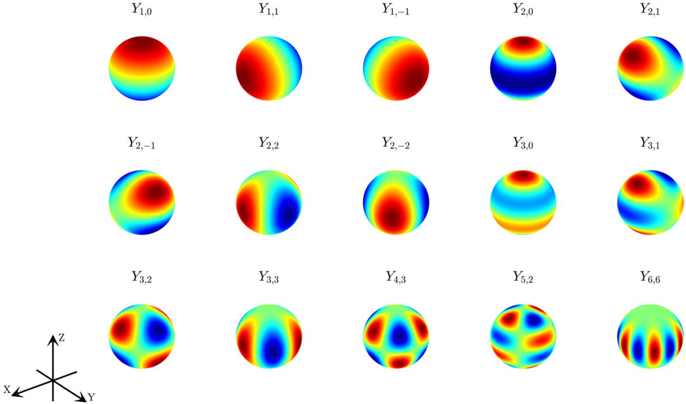
- Multipole moments:

$$M_{lm} = \int_{\mathbb{R}^3} \rho(\boldsymbol{x}) \, r^l \, Y_{lm}^* \, \mathrm{d}^3\boldsymbol{x}$$

Field multipoles in Newtonian gravity



Angular structure encoded into spherical harmonics $Y_{lm}(\theta, \phi)$

Geroch-Hansen multipole moments

- Consider a **stationary** spacetime with **timelike** Killing field ξ^a

[Geroch, JMP **11** (1970) 2580 | Hansen, JMP **15** (1974) 46]

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$$\lambda \equiv -\xi^a \xi_a \qquad \text{and} \qquad \omega_a \equiv \varepsilon_{abcd} \xi^b \nabla^c \xi^d$$

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- Define a **complex potential** $\Phi = \Phi_M + \mathrm{i}\Phi_S$, with

$$\Phi_M \equiv \frac{1 - \lambda^2 - \omega^2}{4\lambda} \qquad \text{and} \qquad \Phi_S \equiv \frac{\omega}{2\lambda}$$

$\hookrightarrow$ relativistic generalization of the Newtonian potential U

[Geroch, JMP **11** (1970) 2580 | Hansen, JMP **15** (1974) 46]

Geroch-Hansen multipole moments

Idea: define **mass-type** and **current-type** multipoles M_{lm} and S_{lm} from the potential Φ at **spatial infinity** i^0

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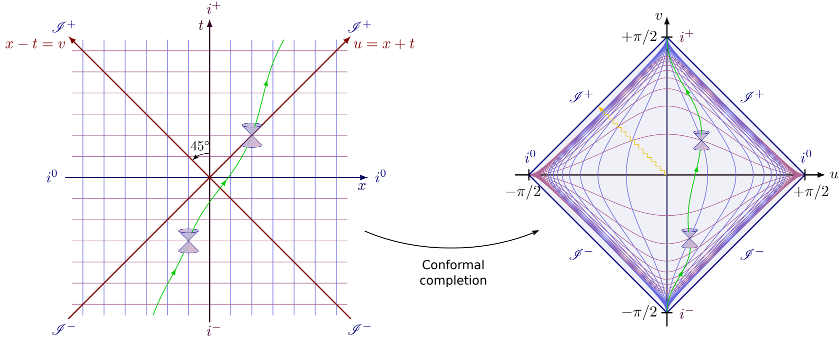

Physical spacetime **Unphysical spacetime**

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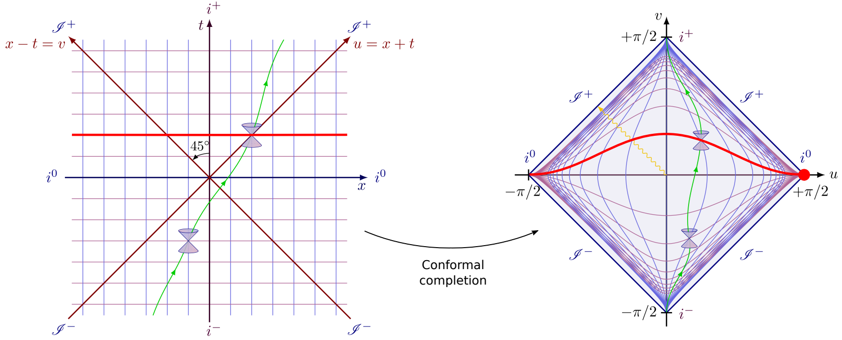

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Field multipoles of a Kerr black hole

- For a Kerr black hole of mass M and spin $S = Ma = M^2\chi$, the norm and twist read (using BL coordinates)

$$\lambda = \frac{r^2 - 2Mr + a^2\cos^2\theta}{r^2 + a^2\cos^2\theta} \qquad \text{and} \qquad \omega = \frac{2Ma\cos\theta}{r^2 + a^2\cos^2\theta}$$

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- Due to the **axisymmetry** of the Kerr metric, the non-zero multipoles all have $m = 0$. They read

$$\boxed{M_l + \mathrm{i}S_l = M(\mathrm{i}a)^l} \quad \iff \quad Z_l = (\mathrm{i}\chi)^l$$

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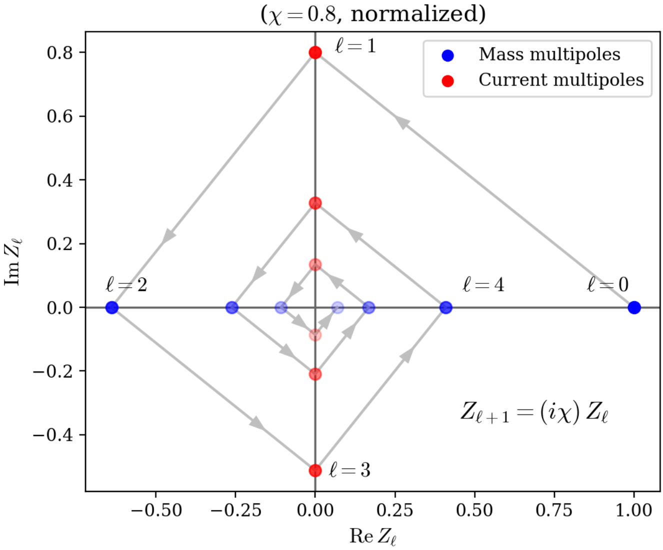
Remark: the Kerr black hole is not merely **oblate**; it is also infinitely structured, yet completely **rigid**

[Hansen, JMP **15** (1974) 46]

Field multipoles of a Kerr black hole

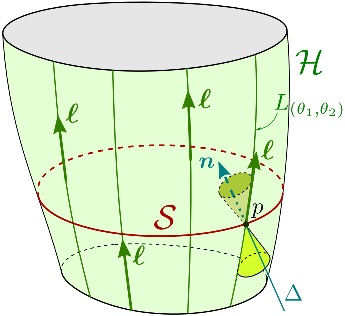


Non-expanding horizons (NEHs)

Definition

A **null hypersurface** $\mathcal{H}$ is a non-expanding horizon iif:

- $\mathcal{H} \sim \mathbb{R} \times \mathbb{S}^2$
- $\forall \boldsymbol{\ell} : \theta_{(\boldsymbol{\ell})} = 0$



[Ashtekar, Fairhurst & Krishnan, PRD **62** (2000) 104025]

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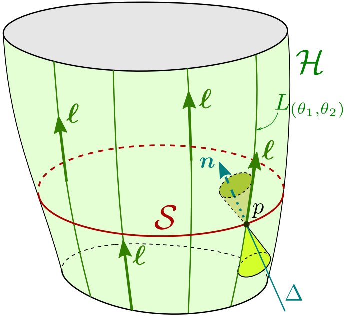


Consequences

- Raychaudhuri equation implies **vanishing shear** of $\boldsymbol{\ell}$: $\sigma_{ab} = 0$
- Induced geometry on $\mathcal{H}$ invariant along $\boldsymbol{\ell}$: $\mathcal{L}_{\boldsymbol{\ell}}\gamma_{ab} = \mathcal{L}_{\boldsymbol{\ell}}\varepsilon_{ab} = 0$

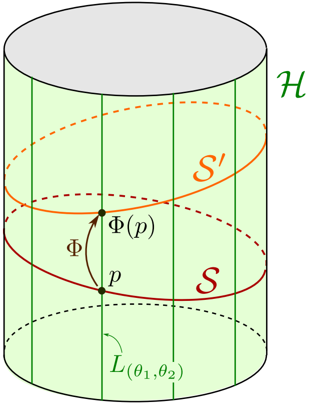
[Ashtekar, Fairhurst & Krishnan, PRD **62** (2000) 104025]

Non-expanding horizons (NEHs)

Main properties

- All cross-sections are **isometric**
  $\hookrightarrow$ well-defined surface area

$$A \equiv \oint_{\mathcal{S}} \boldsymbol{\varepsilon}$$



[Ashtekar, Beetle & Lewandowski, CQG **19** (2002) 1195]

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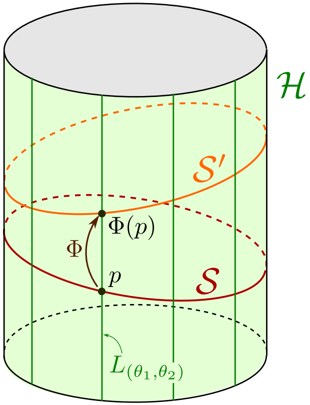
- **Rotation 1-form** ω_a defined by

$$\mathcal{D}_a \ell^b \stackrel{\mathcal{H}}{=} \omega_a \ell^b,$$

  with $\mathcal{D}_a$ the **intrinsic connection** induced on $\mathcal{H}$



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Coulomb-type Weyl curvature scalar

- Given a complex null tetrad $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \boldsymbol{m}^*)$, the Coulomb-type curvature information is encoded into the **Weyl scalar**

$$\Psi_2 \equiv C_{abcd}\ell^a m^b m^{*c} n^d$$

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- On a NEH $\mathcal{H}$, this curvature scalar is **frame-invariant**, with

$$\begin{aligned} \mathrm{Re}\,\Psi_2 &\stackrel{\mathcal{H}}{=} -\tfrac{1}{4}\,\mathcal{R} \quad &\longrightarrow \quad &\textbf{shape} \text{ (intrinsic curvature)} \\ (\mathrm{Im}\,\Psi_2)\,\varepsilon_{ab} &\stackrel{\mathcal{H}}{=} \mathcal{D}_{[a}\omega_{b]} \quad &\longrightarrow \quad &\textbf{rotational structure} \text{ (vorticity)} \end{aligned}$$

$\hookrightarrow$ natural candidate for horizon multipoles on $\mathcal{S}$

[Ashtekar, Beetle & Lewandowski, CQG **19** (2002) 1195]

Effective sources for horizon multipoles

	Electrostatics	Horizon (shape)
Volume	$Q_{lm} = \int_{\mathbb{R}^3} r^\ell \rho \, Y_{lm} \, \mathrm{d}V$	
Surface	$Q_{lm} \propto \oint_{\mathbb{S}^2} \tilde{\rho} \, Y_{lm} \, \mathrm{d}S$	$I_{lm} \propto \oint_{\mathcal{S}} \mathcal{R} \, Y_{lm} \, \mathrm{d}S$
Source	Surface charge density $\tilde{\rho}$	Intrinsic curvature $\mathcal{R}$

thin shell

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	Magnetostatics	Horizon (current)
Volume	$\mu_{lm} = \int_{\mathbb{R}^3} r^\ell \varepsilon^{AB} j_B\, D_A Y_{lm}\, \mathrm{d}V$	
Surface	$\mu_{lm} \propto \oint_{\mathbb{S}^2} \varepsilon^{AB} D_A \tilde{j}_B\, Y_{lm}\, \mathrm{d}S$	$L_{lm} \propto \oint_{\mathcal{S}} \varepsilon^{AB} D_A \omega_B\, Y_{lm}\, \mathrm{d}S$
Source	Curl of surface current $\tilde{j}_A$	Curl of rotation 1-form ω_A

Horizon multipole moments of a NEH

Definition

The **shape** and **current** multipole moments I_{lm} and L_{lm} are defined by

$$I_{lm} + \mathrm{i} L_{lm} \equiv -\oint_{\mathcal{S}} \Psi_2 \, \mathring{Y}_{lm} \, \mathrm{d}S$$

[Ashtekar *et al.*, CQG **21** (2004) 2549 | Ashtekar *et al.*, JHEP **2022** (2022) 28]

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- Since all the horizon cross-sections are isometric, this definition is **independent** of the choice of $\mathcal{S}$

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Properties

- Since all the horizon cross-sections are isometric, this definition is **independent** of the choice of $\mathcal{S}$
- The spherical harmonics $\mathring{Y}_{lm}$ are defined **geometrically** from canonical polar coordinates (ϑ, φ)

[Ashtekar *et al.*, CQG **21** (2004) 2549 | Ashtekar *et al.*, JHEP **2022** (2022) 28]

Horizon multipole moments of a NEH

Problem: define canonical spherical harmonics on $\mathcal{S}$

Axisymmetric NEH

Symmetry $\rightarrow$ **axial Killing vector** $\rightarrow$ canonical (ϑ, φ)

Generic NEH

No symmetry $\rightarrow$ **conformal map** to $\mathbb{S}^2$ $\rightarrow$ canonical (ϑ, φ)

[Ashtekar *et al.*, CQG **21** (2004) 2549 | Ashtekar *et al.*, JHEP **2022** (2022) 28]

Horizon multipoles of a Kerr black hole

- Due to the **axisymmetry** of the Kerr metric, all non-zero multipoles have $m = 0$. The **reflection symmetry** accross the equatorial plane further implies

$$\forall n \in \mathbb{N}, \quad I_{2n+1} = 0 \quad \text{and} \quad L_{2n} = 0$$

[Gourgoulhon, Le Tiec & Casals, to appear in PRD (2026), gr-qc/2602.05823]

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- Moreover, in the regime of **small-spin** values $0 < \chi \ll 1$, the non-zero horizon multipoles behave as

$$\boxed{I_l + \mathrm{i} L_l \sim \alpha_l \, (\mathrm{i}\chi)^l}$$

$\hookrightarrow$ **same scaling** as normalized field multipoles $Z_l = (\mathrm{i}\chi)^l$

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Horizon multipoles of a Kerr black hole

- **Axisymmetric definition:** closed-form expression $(\hat{a} \equiv a/r_+)$

$$I_l^{\rm axi} + \mathrm{i} L_l^{\rm axi} = 2^\ell \alpha_l \, (\mathrm{i}\hat{a})^l \, {}_2F_1\left(\tfrac{l}{2}, \tfrac{l-1}{2}, l + \tfrac{3}{2}; -\hat{a}^2\right)$$

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- **Generic definition:** no closed-form $\rightarrow$ numerical evaluation
- Nevertheless, the conformal factor is known analytically
$\hookrightarrow$ full geometry under analytic control

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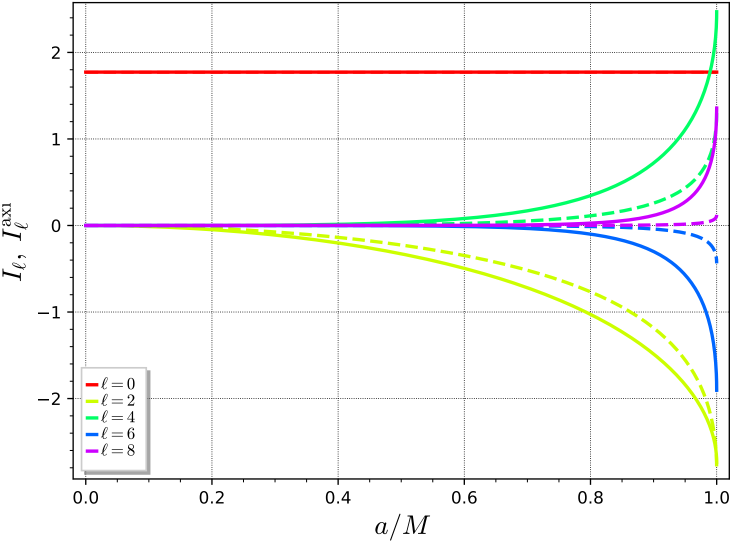
Shape multipoles of a Kerr black hole



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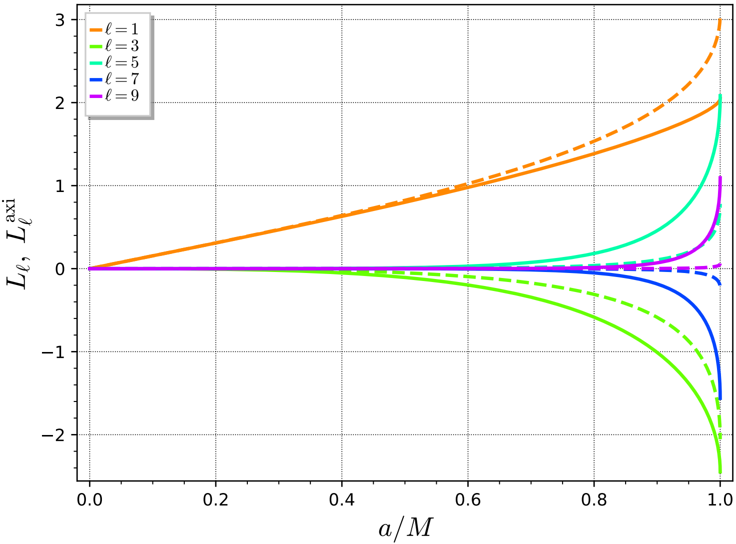
Current multipoles of a Kerr black hole



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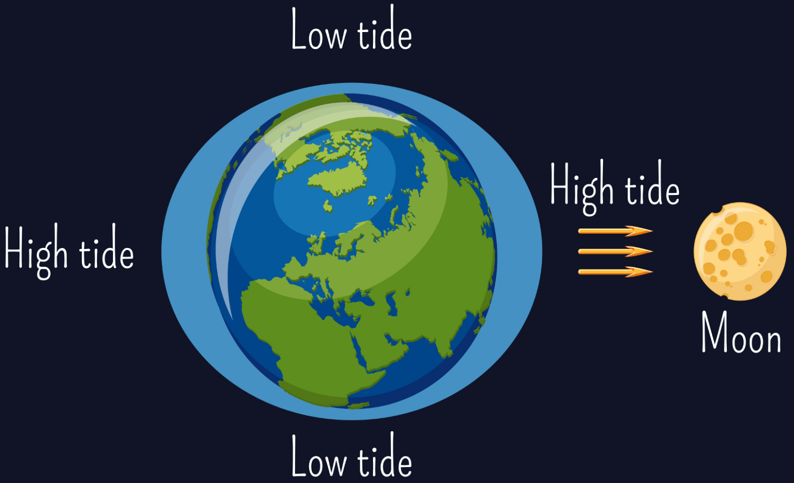
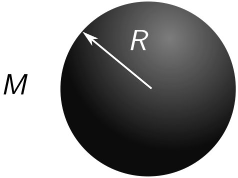
TIDES

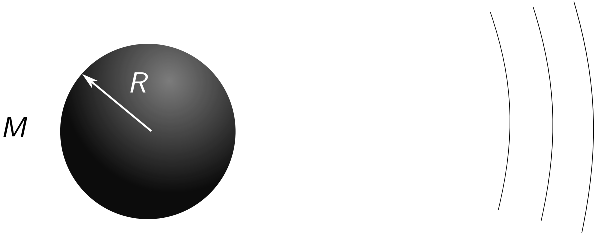


Newtonian theory of Love numbers



$$U = \frac{M}{r}$$

Newtonian theory of Love numbers



$$\mathcal{E}_{ab} = -\partial_{\langle a}\partial_{b\rangle} U_{\text{ext}}(\mathbf{0})$$

$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab}$$

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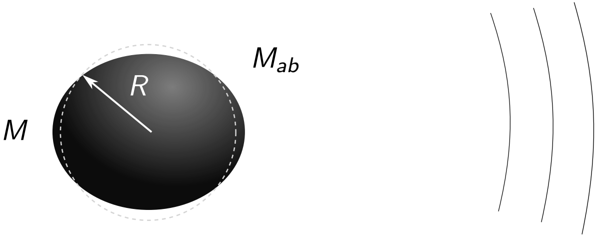
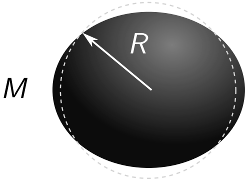


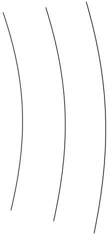
$$\mathcal{E}_{ab} = -\partial_{\langle a}\partial_{b\rangle} U_{\mathrm{ext}}(\mathbf{0})$$

$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} + \frac{3}{2} \frac{x^a x^b}{r^5} M_{ab}$$

Newtonian theory of Love numbers



$$M_{ab} = \lambda_2 \mathcal{E}_{ab}$$



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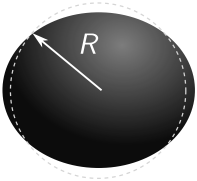
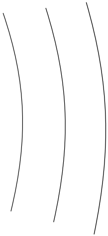
Newtonian theory of Love numbers

M



$$M_{ab} = \lambda_2 \mathcal{E}_{ab}$$
$$= -\tfrac{2}{3} k_2 R^5 \mathcal{E}_{ab}$$



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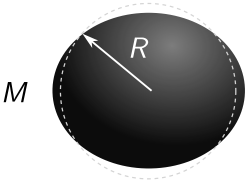

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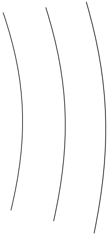
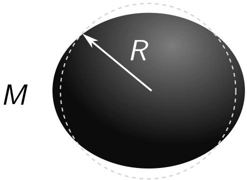


$$\mathcal{E}_{ab} = -\partial_{\langle a}\partial_{b\rangle} U_{\text{ext}}(\mathbf{0})$$

$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} \left[1 + 2 k_2 \left(\frac{R}{r} \right)^5 \right]$$

Newtonian theory of Love numbers



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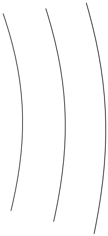

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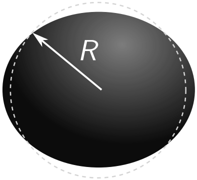
$$U = \frac{M}{r} - \sum_{l \geqslant 2} \frac{(l-2)!}{l!}\, x^L \mathcal{E}_L \left[1 + 2 k_l \left(\frac{R}{r} \right)^{2l+1} \right]$$

Tidal Love numbers k_l encode the body's **internal structure**

Newtonian theory of Love numbers

M



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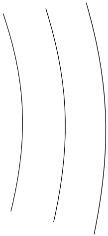

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Tidal Love numbers k_l encode the body's **internal structure**

Relativistic theory of Love numbers

- **Tidal** vs **response** decomposition:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \underbrace{h^{\text{tidal}}_{\alpha\beta}}_{\sim r^{l}} + \underbrace{h^{\text{resp}}_{\alpha\beta}}_{\sim r^{-(l+1)}}$$

[Binnington & Poisson, PRD **80** (2009) 084018 | Damour & Nagar, PRD **80** (2009) 084035]

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- *Nonspinning* BHs have **vanishing** static Love numbers
- Effects of **spin**:
  - coupling between multipoles & m-dependence
  - mixing between electric and magnetic sectors

$\hookrightarrow$ **anisotropic** and **mode-coupled** response

[Binnington & Poisson, PRD **80** (2009) 084018 | Damour & Nagar, PRD **80** (2009) 084035]

Love numbers of a Kerr black hole

- Love numbers computed to **linear order** in spin $\chi \equiv S/M^2$

[Le Tiec & Casals, PRL **126** (2021) 131102 | Le Tiec, Casals & Franzin, PRD **103** (2021) 084021]

Love numbers of a Kerr black hole

- Love numbers computed to **linear order** in spin $\chi \equiv S/M^2$
- The modes of the mass/current **quadrupole moments** are

$$\delta M_{2m} \doteq \frac{im\chi}{180}\,(2M)^5\,\mathcal{E}_{2m} \quad \text{and} \quad \delta S_{2m} \doteq \frac{im\chi}{180}\,(2M)^5\,\mathcal{B}_{2m}$$

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- The associated **tidal Love numbers** are

$$\boxed{k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} \doteq -\frac{im\chi}{120}} \quad \text{and} \quad \boxed{k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} \doteq 0}$$

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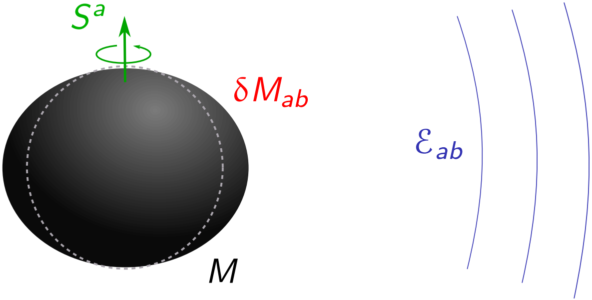
- For a black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \sim 10^{-3} \quad \longrightarrow \quad \textbf{extremely small response}$$

[Le Tiec & Casals, PRL **126** (2021) 131102 | Le Tiec, Casals & Franzin, PRD **103** (2021) 084021]

Geometric interpretation

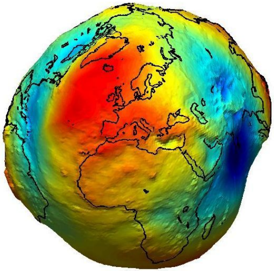


$$\delta M_{ab} \propto M^3\, S^c\, \mathcal{E}^d{}_{\langle a}\, \varepsilon_{b\rangle cd} \quad \longrightarrow \quad \text{tidal bulge } \textbf{rotated by } 45^\circ$$

[Le Tiec & Casals, PRL **126** (2021) 131102 | Le Tiec, Casals & Franzin, PRD **103** (2021) 084021]

Do isolated black holes have hair?

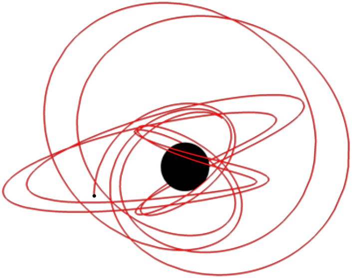
Geodesy



M_{lm} arbitrary

rich internal structure

Black holes



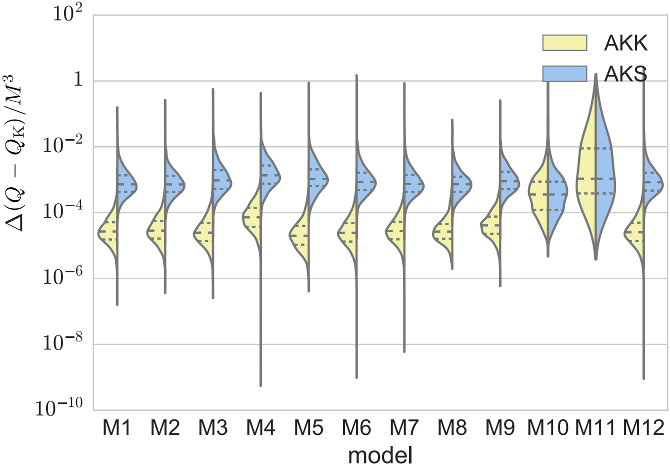
$$M_{l0} + \mathrm{i}S_{l0} = M(\mathrm{i}a)^l$$

completely fixed by (M, a)

Objective: measure the *multipolar structure* of astrophysical BHs

LISA constraints from EMRIs

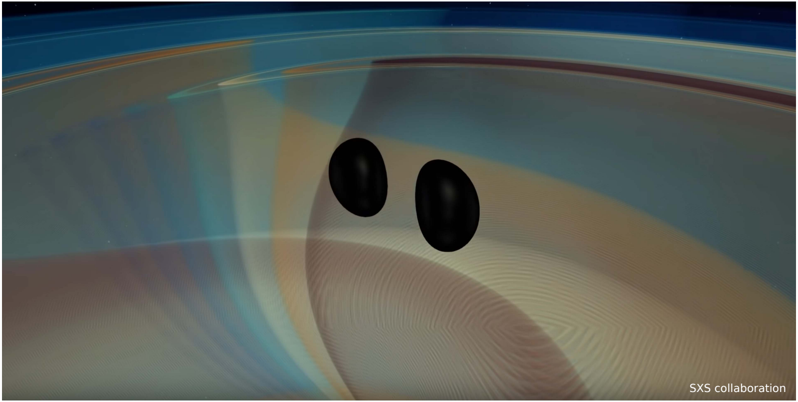
Waveform $\rightarrow$ measure (M_{lm}, S_{lm}) $\rightarrow$ test $Z_l = (i\chi)^l$



[Babak, Gair, Sesana *et al.*, PRD **95** (2017) 103012]

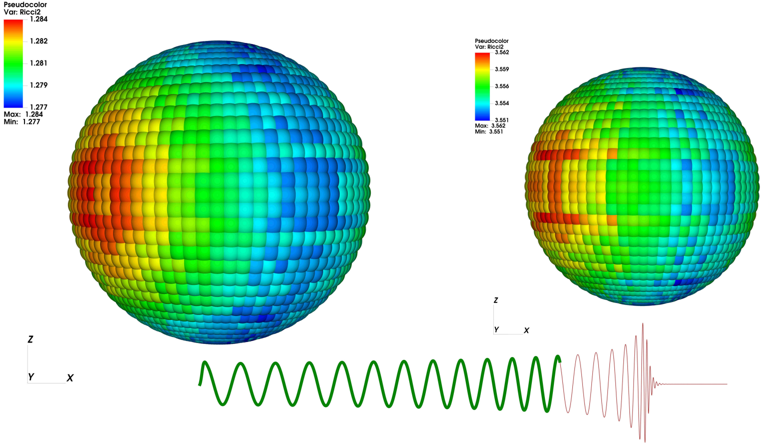
Do tidally-interacting black holes deform?

Isolated BH: **no hair** $\longrightarrow$ Binary BH: **do they respond?**



Objective: measure the *dynamical response* of astrophysical BHs

Black hole tomography by GW observations



Waveform $\rightarrow$ measure (I_{lm}, L_{lm}) $\rightarrow$ Love numbers

[Prasad, PRD **109** (2024) 044033 | Ribes Metidieri *et al.*, PRD **111** (2025) 104075]

Take-home messages

1. Horizon multipoles provide a **geometric characterization** of black hole shape
2. Kerr black holes have a **purely dissipative** (no conservative) tidal response
3. Gravitational-wave observations enable **precision tests** of:
   - the no-hair theorem (stationary geometry)
   - tidal deformability (dynamical response)
4. **Optical shape** (black hole shadow) provides a complementary observable, offering an independent probe of geometry

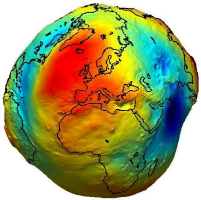
From geometry to observable signatures

Additional Material

Field multipoles in Newtonian gravity

- Gravitational field:

$$U(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3\boldsymbol{x}'$$



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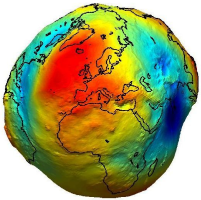

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- Multipole expansion:

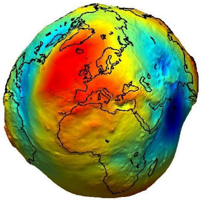
$$U(\boldsymbol{x}) = \sum_{l\geqslant 0} \frac{(2l-1)!!}{l!}\,\frac{n^{\langle L\rangle}}{r^{l+1}}\,M_L = \frac{M}{r} + \frac{3}{2}\,\frac{n^{\langle a}n^{b\rangle}}{r^3}\,M_{ab} + \cdots$$

Notations:

multi-index $L \equiv a_1 \cdots a_l$

symmetric trace-free part $\langle\cdot\rangle$

Field multipoles in Newtonian gravity



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- Multipole moments:

$$M_L = \int_{\mathbb{R}^3} \rho(\boldsymbol{x})\,x_L\,\mathrm{d}^3\boldsymbol{x}$$

Notations:

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symmetric trace-free part $\langle\cdot\rangle$

Relativistic theory of Love numbers

- Electric-type and magnetic-type tidal moments:

$$\mathcal{E}_L \propto [C_{0a_1 0a_2;a_3\ldots l}]^{\mathrm{STF}} \quad \text{and} \quad \mathcal{B}_L \propto [\varepsilon_{a_1 bc} C_{a_2 0bc;a_3\cdots a_l}]^{\mathrm{STF}}$$

- Metric and Geroch-Hansen multipole moments:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \underbrace{h^{\mathrm{tidal}}_{\alpha\beta}}_{\sim r^l} + \underbrace{h^{\mathrm{resp}}_{\alpha\beta}}_{\sim r^{-(l+1)}} \quad \longrightarrow \quad \begin{cases} M_L = \bar{M}_L + \delta M_L \\ S_L = \bar{S}_L + \delta S_L \end{cases}$$

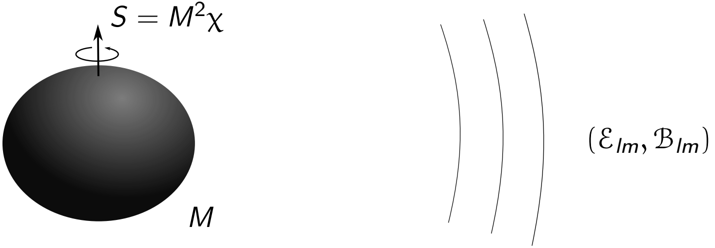
- Two families of tidal deformability parameters:

$$\delta M_L = \lambda_l^{\mathrm{el}} \mathcal{E}_L \quad \text{and} \quad \delta S_L = \lambda_l^{\mathrm{mag}} \mathcal{B}_L$$

- Dimensionless tidal Love numbers:

$$k_l^{\mathrm{el/mag}} \equiv -\frac{(2l-1)!!}{2(l-2)!} \frac{\lambda_l^{\mathrm{el/mag}}}{R^{2l+1}}$$

Investigating Kerr's Love



Computational pipeline

$$(\mathcal{E}_{lm}, \mathcal{B}_{lm}) \to \Psi_0 \to h_{\alpha\beta} \to (M_{lm}, S_{lm}) \to \lambda_{lm}^{M/S,\mathcal{E}/\mathcal{B}}$$

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c\to\infty} \Psi_0^{lm} \propto \mathcal{E}_{lm}\, r^{\ell-2} \left[1 + 2k_{lm}\,(R/r)^{2\ell+1}\right] {}_2Y_{lm}(\theta, \phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\Psi_0^{lm} \propto \left[\mathcal{E}_{lm} + \tfrac{3i}{\ell+1}\,\mathcal{B}_{lm}\right] R_{lm}(r)\; {}_2Y_{lm}(\theta, \phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{lm}(r) = \underbrace{r^{\ell-2}\,(1 + \cdots)}_{\text{tidal field } R_{lm}^{\text{tidal}}} + k_{lm}\, \underbrace{r^{-\ell-3}\,(1 + \cdots)}_{\text{linear response } R_{lm}^{\text{resp}}}$$

Kerr black hole linear response

$$R_{lm}(r) = \underbrace{R_{lm}^{\text{tidal}}(r)}_{\sim r^{\ell-2}} + 2k_{lm}\underbrace{R_{lm}^{\text{resp}}(r)}_{\sim r^{-(\ell+3)}}$$

- The coefficients k_{lm} can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$$k_{lm} = -im\chi \frac{(\ell+2)!(\ell-2)!}{4(2\ell+1)!(2\ell)!} \prod_{n=1}^{\ell} \left[n^2(1-\chi^2) + m^2\chi^2\right]$$

- The linear response vanishes identically when:
  - the black hole spin vanishes ($\chi = 0$)
  - the tidal field is axisymmetric ($m = 0$)
- Reconstruct the Kerr black hole response $h_{\alpha\beta}^{\text{resp}}$ via Ψ^{resp}

Why analytic continuation?

$$R_{lm}(r) = \underbrace{r^{\ell-2}\,(1+\cdots)}_{\text{tidal field } R_{lm}^{\text{tidal}}} + k_{lm}\,\underbrace{r^{-\ell-3}\,(1+\cdots)}_{\text{linear response } R_{lm}^{\text{resp}}}$$

Ambiguity in the linear response [Fang & Lovelace 2005; Gralla 2018]

The decaying solution R_{lm}^{resp} is affected by a radial coord. transfo.

Ambiguity in the tidal field [Pani, Gualtieri, Maselli & Ferrari 2015]

The growing solution $R_{lm}^{\text{tidal}} + \alpha R_{lm}^{\text{resp}}$ still qualifies as a tidal solution

Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\mathrm{el}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\mathrm{mag}} \mathcal{B}_{ab}$$

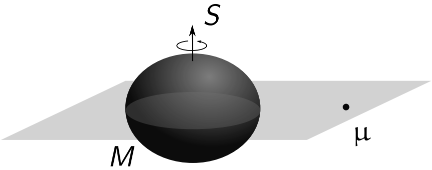
- For a spinning black hole we have the more general tensorial relations

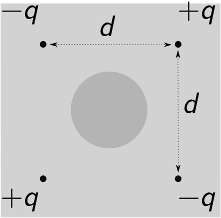
$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}^{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}^{cd}$$

- To *linear order* in the black hole spin vector S^a we find

$$\delta M_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{E}^d{}_{(a} \varepsilon_{b)cd}$$
$$\delta S_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{B}^d{}_{(a} \varepsilon_{b)cd}$$

Newtonian static quadrupolar tide

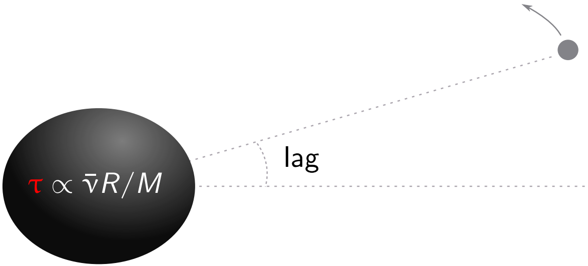




$$\mathcal{E}_{ab} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

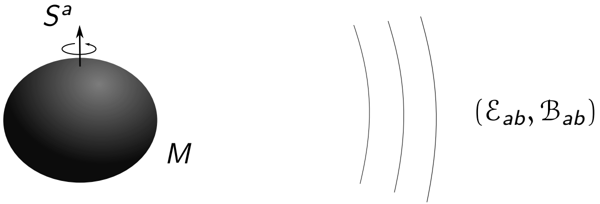
$$\delta M_{ab} \doteq \underset{\substack{\uparrow \\ \frac{\chi}{180}\,(2M)^5\,\frac{\mu}{r^3} \,=\, qd^2}}{3Q} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Tidal dissipation: lag, heating and torquing



$$M_{ab}(t) = -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t) - \tau \dot{\mathcal{E}}_{ab}(t) + \cdots \right]$$
$$= -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t - \tau) + \cdots \right]$$

Tidal torquing of a spinning black hole



- Any spinning body interacting with a tidal environment suffers a tidal torquing [Thorne & Hartle 1980]

$$\langle \dot{S}^a \rangle = -\varepsilon^{abc} \langle M_{bd} \mathcal{E}^d{}_c + S_{bd} \mathcal{B}^d{}_c \rangle$$

- Applied to a spinning black hole this yields

$$\langle \dot{S} \rangle \doteq -\frac{8}{45} M^5 \chi \left[2\langle \mathcal{E}^{ab} \mathcal{E}_{ab} \rangle - 3 \langle \mathcal{E}_{ab} s^b \mathcal{E}^{ac} s_c \rangle + (\mathcal{E} \to \mathcal{B}) \right]$$

- Full agreement with independant calculation by [Poisson 2004]

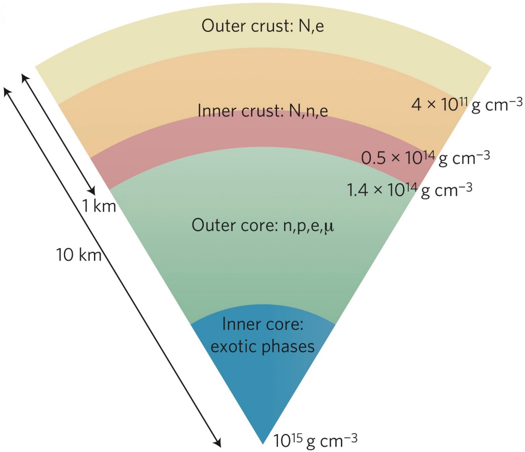
Do black holes have *zero* Love numbers?

Reference	**Background**	**Tidal field**
[Binnington & Poisson 2009]	Schwarzschild	weak, generic l
[Damour & Nagar 2009]	Schwarzschild	weak, generic l
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti *et al.* 2013]	Schwarzschild	weak, electric, $l = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani *et al.* 2015]	Kerr to $O(S^2)$	weak, $(lm) = (20)$
[Le Tiec & Casals 2021]	Exact Kerr	weak, generic l
[Chia 2021]	Exact Kerr	weak, generic l
[Goldberger *et al.* 2021]	Exact Kerr	weak, generic l
[Charalambous *et al.* 2021]	Exact Kerr	weak, generic l

A burst of activity on BH tidal deformability

- Other backgrounds, generic spin-s fields and higher dimensions
  [Hui *et al.*, JCAP 2021; Pereñiguez & Cardoso, PRD 2022; Rodriguez *et al.*, PRD 2023; Charalambous & Ivanov, JHEP 2023; Charalambous, JHEP 2024]
- Dissipative nature of Kerr black hole tidal deformability
  [Chia, PRD 2021; Goldberger *et al.*, JHEP 2021; Charalambous, JHEP 2021; Prasad Bhatt *et al.*, PRD 2023]
- Hidden symmetry and vanishing black hole Love numbers
  [Charalambous *et al.*, PRL 2021; Hui *et al.*, JCAP 2022; Charalambous *et al.*, JHEP 2022; Achour *et al.*, JHEP 2022; Hui *et al.*, JHEP 2022; Berens *et al.*, JCAP 2023; Katagiri *et al.*, PRD 2023; Rai & Santoni 2024]
- Scattering amplitudes and vanishing black hole Love numbers
  [Creci *et al.*, PRD 2021; Ivanov & Zhou, PRL 2023; Saketh *et al.*, PRD 2024]
- Effective Field Theory, matching and logarithmic corrections
  [Ivanov & Zhou, PRD 2023]
- Nonlinearities in the tidal Love numbers of black holes
  [De Luca *et al.*, PRD 2023; Maria Riva *et al.* 2023; Hadad *et al.* 2024; Combaluzier-Szteinsznaider *et al.* 2024; Kehagias & Riotto 2024]

What is the internal structure of a neutron star?

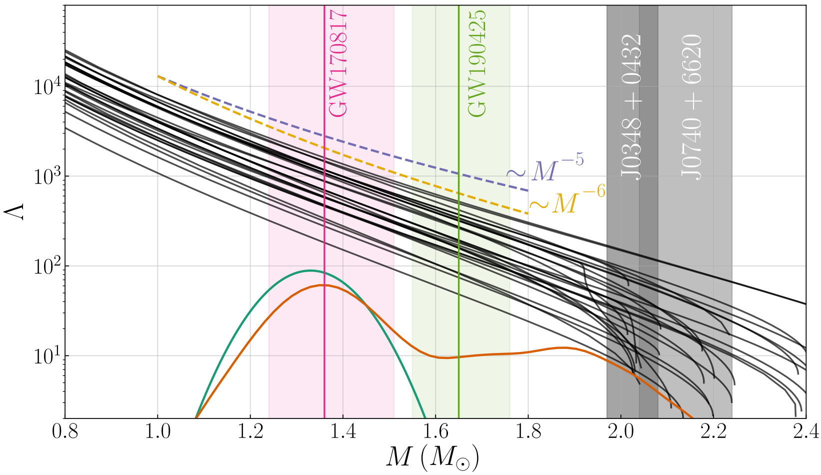


GW observations as probes of **neutron star equation of state**

First constraints from GW observations

[Chatziioannou, GRG 2020]

$\Lambda \equiv \lambda_2/M^5 \propto k_2/C^5$



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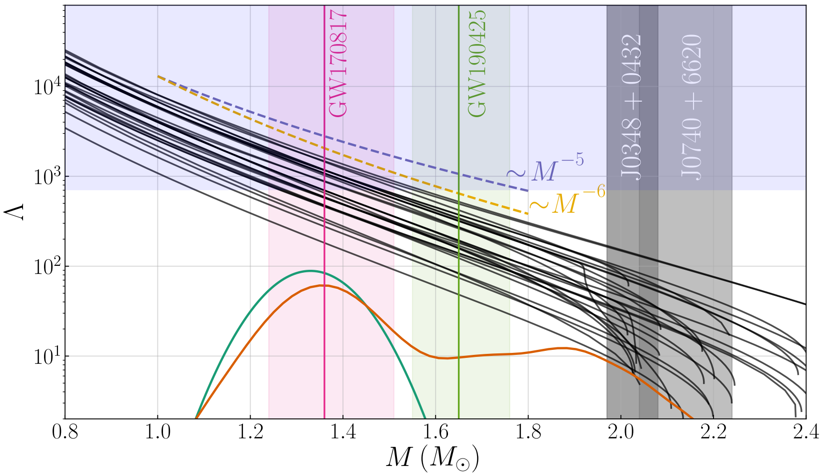
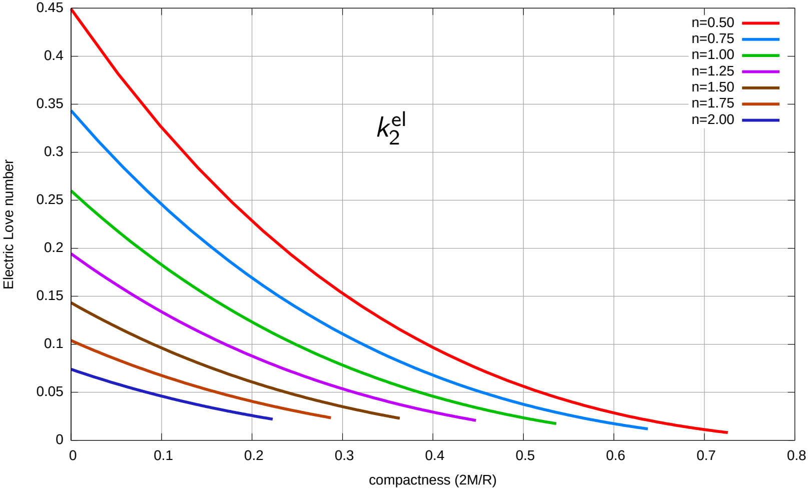


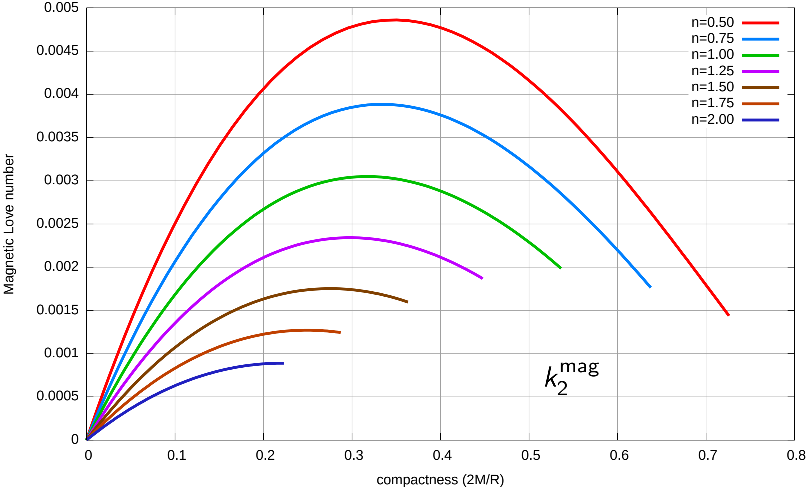
Love numbers of neutron stars

[Binnington & Poisson, PRD 2009]



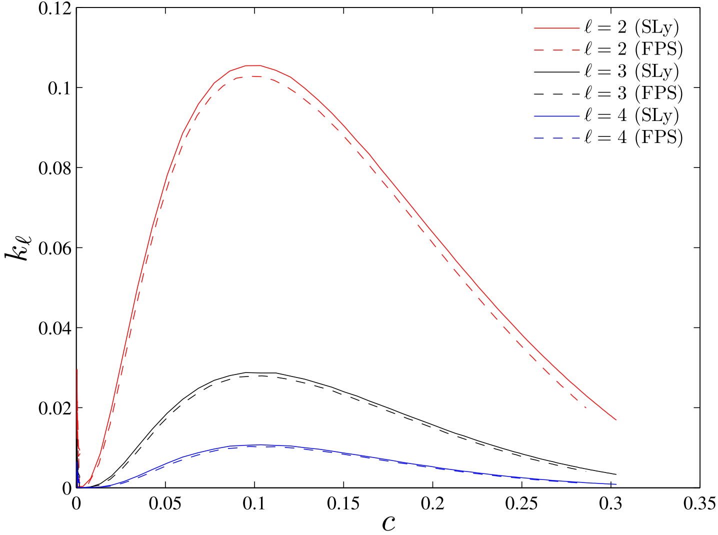
Love numbers of neutron stars

[Binnington & Poisson, PRD 2009]



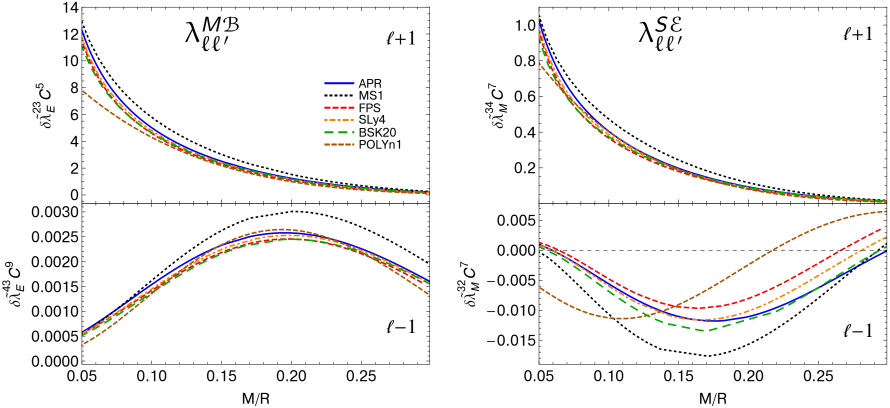
Love numbers of neutron stars

[Damour & Nagar, PRD 2009]



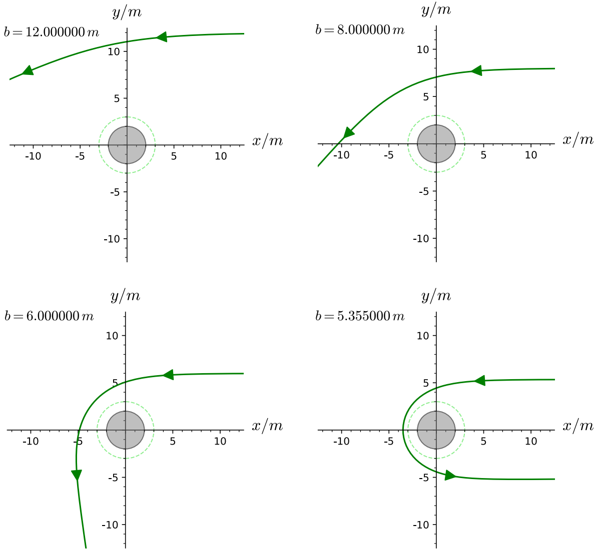
Love numbers of *spinning* neutron stars

[Pani, Gualtieri & Ferrari, PRD 2015]



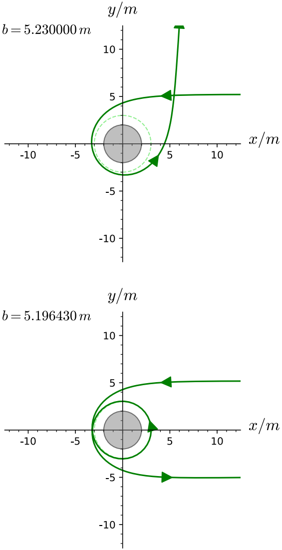
Restricted to an *axisymmetric* tidal perturbation ($m = 0$)

Ray tracing in Schwarzschild spacetime



[Gourgoulhon, in *Geometry and Physics of Black Holes* (2026) | relativite.obspm.fr/blackholes]

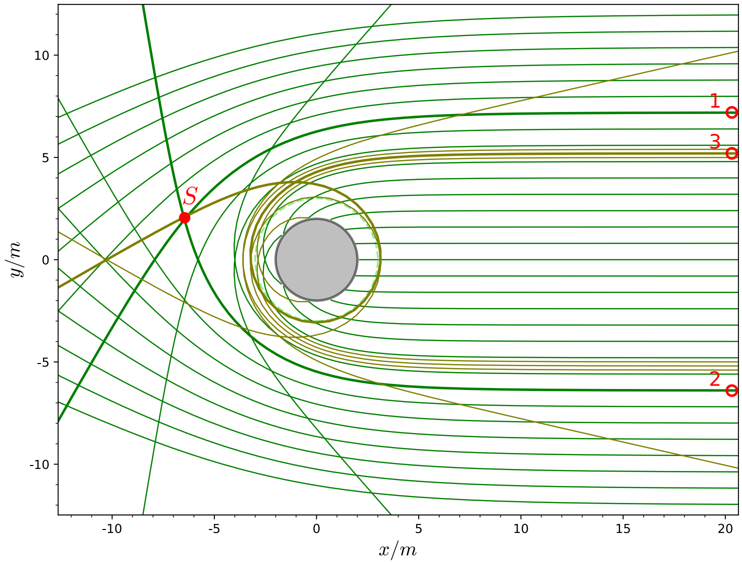
Ray tracing in Schwarzschild spacetime



$b_c = 3\sqrt{3}\,m \simeq 5.196152\,m$

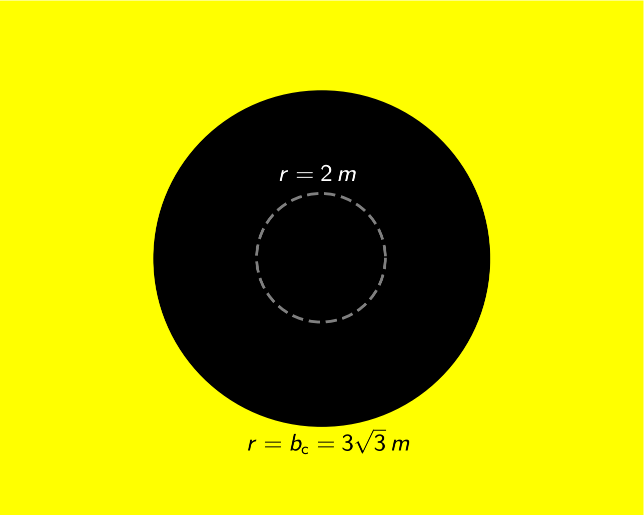
[Gourgoulhon, in *Geometry and Physics of Black Holes* (2026) | relativite.obspm.fr/blackholes]

Ray tracing in Schwarzschild spacetime



[Gourgoulhon, in *Geometry and Physics of Black Holes* (2026) | relativite.obspm.fr/blackholes]

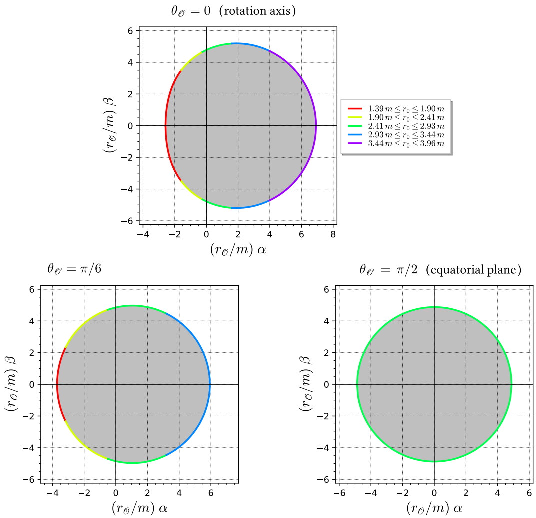
Ray tracing in Schwarzschild spacetime



[Gourgoulhon, in *Geometry and Physics of Black Holes* (2026) | relativite.obspm.fr/blackholes]

Ray tracing in Kerr spacetime

Kerr spin parameter $a = 0.95\,m$



[Gourgoulhon, in *Geometry and Physics of Black Holes* (2026) | relativite.obspm.fr/blackholes]

Ray-tracing in Kerr spacetime

Kerr spin parameter $a = m$

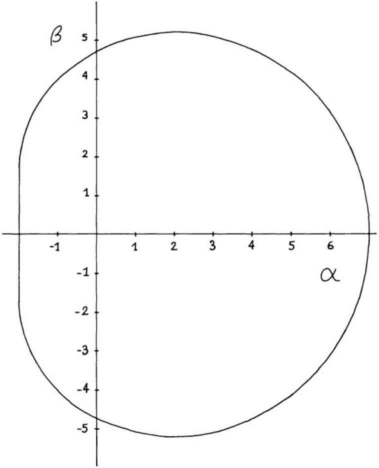
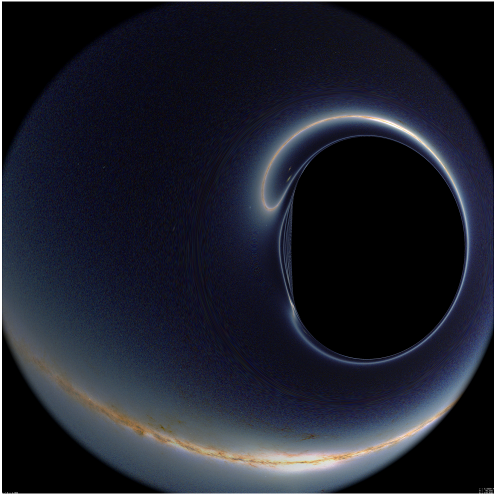


Figure 6. The apparent shape of an extreme ($a = m$) Kerr black hole as seen by a distant observer in the equatorial plane, if the black hole is in front of a source of illumination with an angular size larger than that of the black hole.

[Bardeen, in *Black Holes: Les Astres Occlus* (1973), Gordon and Breach Science Publishers]

Kerr silhouette on a Milky Way background

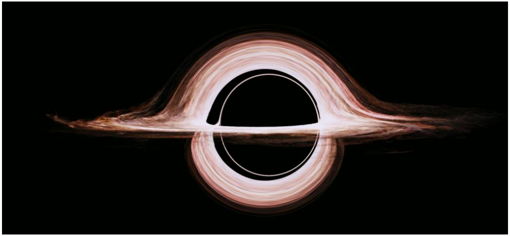
Kerr spin parameter $a = ?\, m$ and $r = 6\, m$

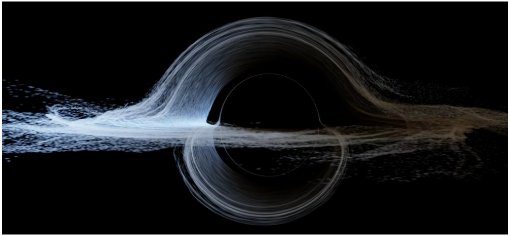


[Riazuelo, IJMPD **29** (2020) 2050109]

Thin accretion disk around a Kerr black hole

Kerr spin parameter $a = 0.6\, m$





+

Light frequency

Doppler shift
Einstein effect

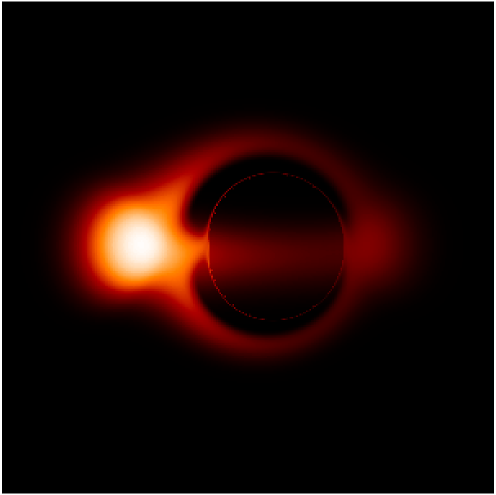
Specific intensity

Liouville theorem

[James, von Tunzelmann, Franklin & Thorne, CQG **32** (2015) 065001]

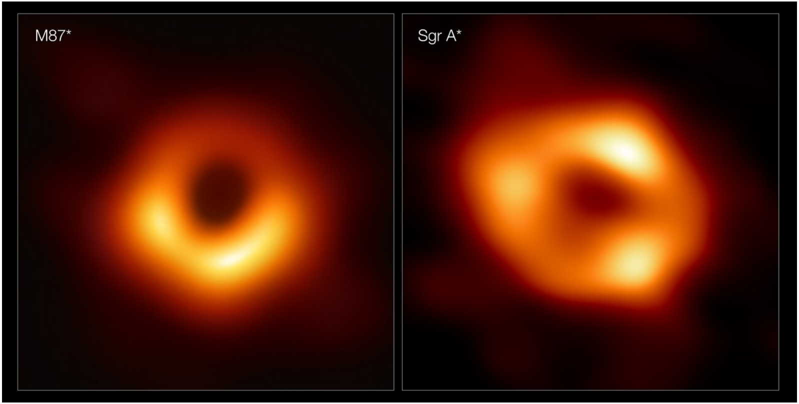
Thick accretion disk around a Kerr black hole

Kerr spin parameter $a = 0.9\, m$



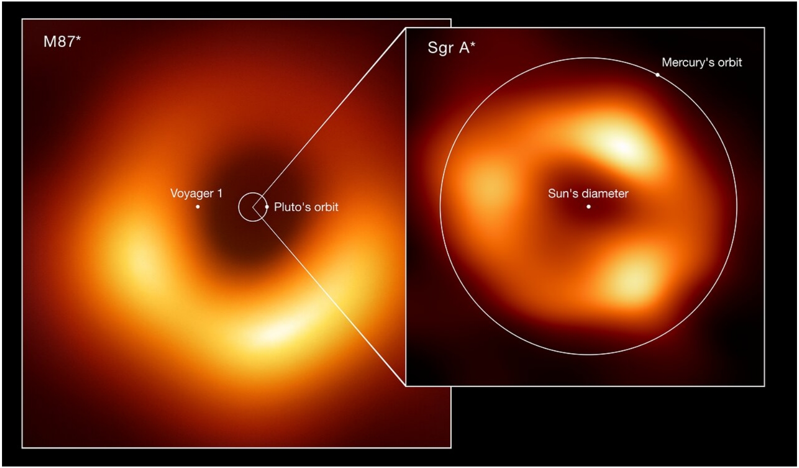
[Vincent, Meliani, Grandclément, Gourgoulhon & Straub, CQG **33** (2016) 105015]

EHT images of M87* and Sgr A*



[Event Horizon Telescope Collaboration, ApJL **875** (2019) L1 | ApJL **930** (2022) L12]

EHT images of M87* and Sgr A*



[Event Horizon Telescope Collaboration, ApJL **875** (2019) L1 | ApJL **930** (2022) L12]