Tidal deformability of spinning black holes

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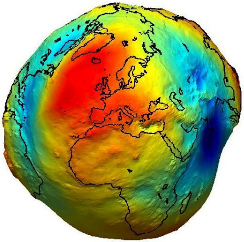
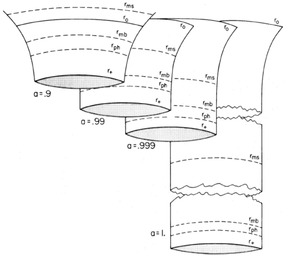
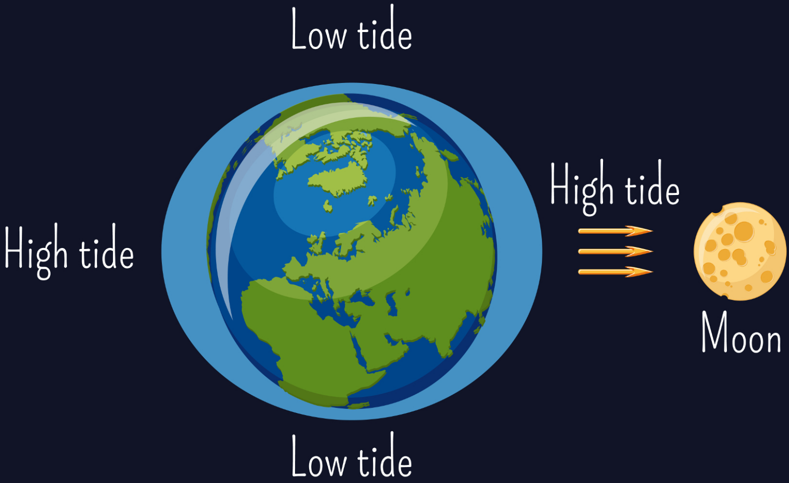




TIDES



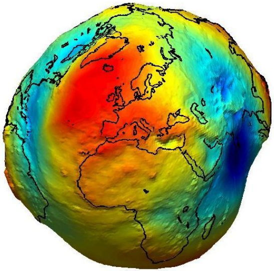
1. Motivations

2. Newtonian tides

3. Black hole Love numbers

Do isolated black holes have hair?

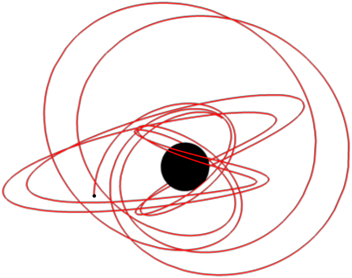
Geodesy



M_{lm} arbitrary

rich internal structure

Black holes



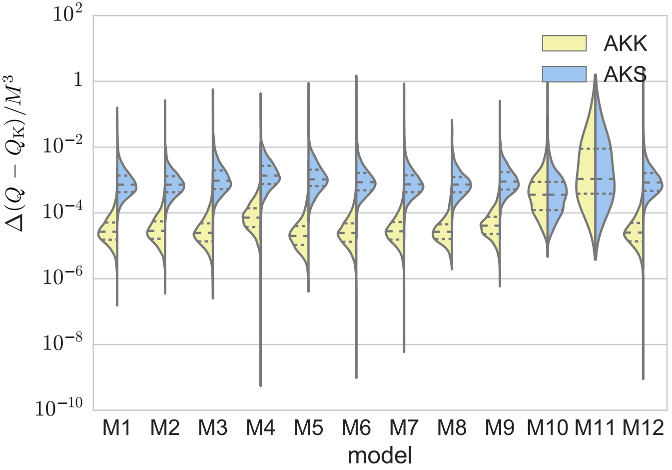
$$M_{l0} + \mathrm{i}S_{l0} = M(\mathrm{i}a)^l$$

completely fixed by (M, a)

Objective: measure the *multipolar structure* of astrophysical BHs

LISA constraints from EMRIs

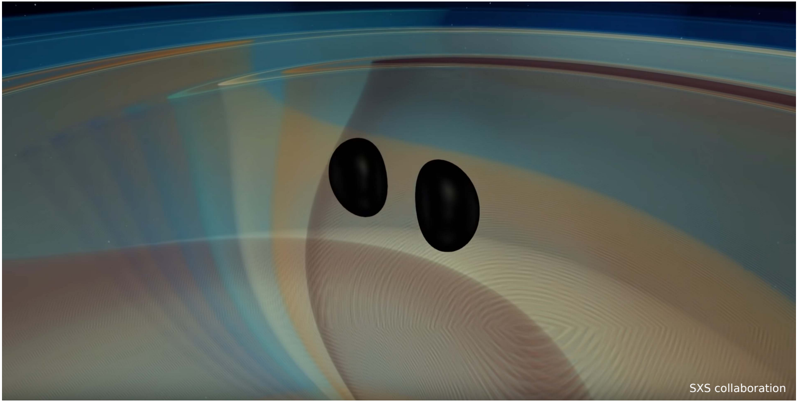
Waveform $\rightarrow$ measure (M_{lm}, S_{lm}) $\rightarrow$ test $Z_l = (i\chi)^l$



[Babak, Gair, Sesana *et al.*, PRD **95** (2017) 103012]

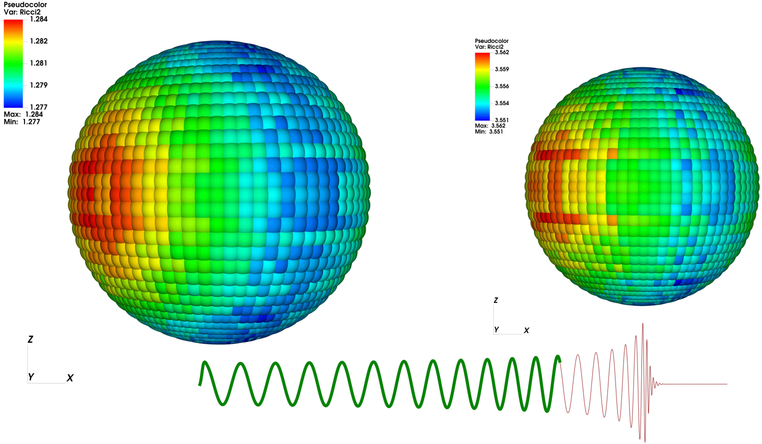
Do tidally-interacting black holes deform?

Isolated BH: **no hair** $\longrightarrow$ Binary BH: **do they respond?**



Objective: measure the *dynamical response* of astrophysical BHs

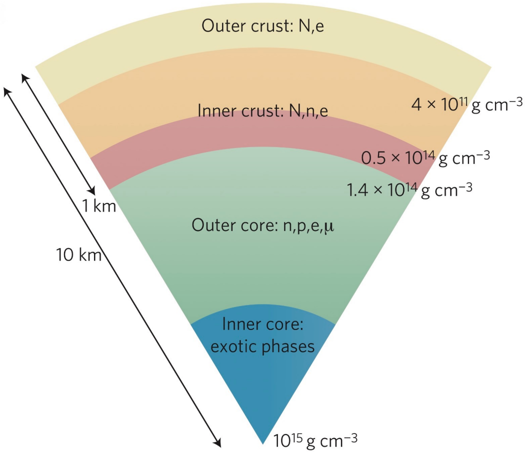
Black hole tomography by GW observations



Waveform $\rightarrow$ measure (I_{lm}, L_{lm}) $\rightarrow$ shape Love numbers

[Prasad, PRD **109** (2024) 044033 | Ribes Metidieri *et al.*, PRD **111** (2025) 104075]

What is the internal structure of a NS?

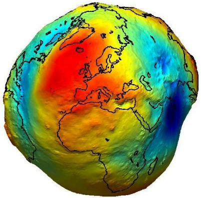


GW observations as probes of **neutron star equation of state**

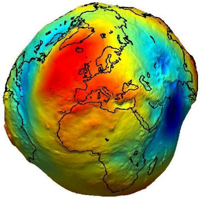
Field multipoles in Newtonian gravity

- Gravitational field:

$$U(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3\boldsymbol{x}'$$



Field multipoles in Newtonian gravity



- Gravitational field:

$$U(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3\boldsymbol{x}'$$

- Multipole expansion:

$$U(r,\theta,\phi) = \sum_{l\geqslant 0}\sum_{|m|\leqslant l} \frac{4\pi}{2l+1}\,\frac{M_{lm}}{r^{l+1}}\,Y_{lm}(\theta,\phi)$$

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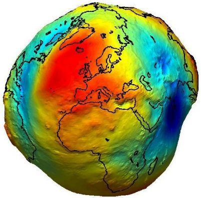

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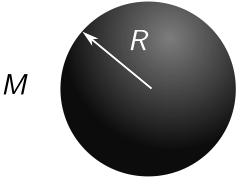
- Multipole moments:

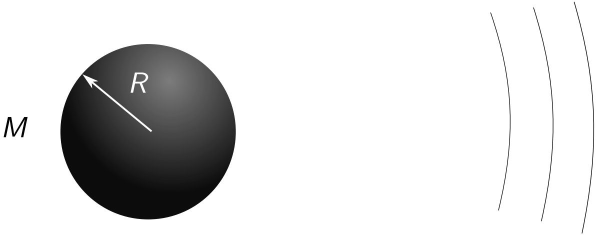
$$M_{lm} = \int_{\mathbb{R}^3} \rho(\boldsymbol{x}) \, r^l \, Y^*_{lm} \, \mathrm{d}^3\boldsymbol{x}$$

Newtonian theory of *field* Love numbers



$$U = \frac{M}{r}$$

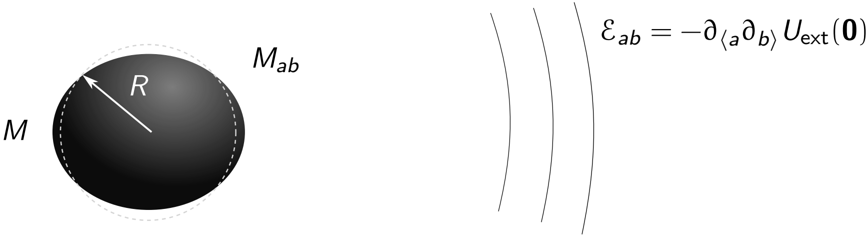
Newtonian theory of *field* Love numbers



$$\mathcal{E}_{ab} = -\partial_{\langle a}\partial_{b\rangle} U_{\rm ext}(\mathbf{0})$$

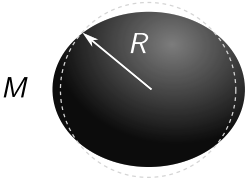
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab}$$

Newtonian theory of *field* Love numbers

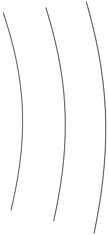


$$U = \frac{M}{r} - \frac{1}{2}x^a x^b \mathcal{E}_{ab} + \frac{3}{2}\frac{x^a x^b}{r^5} M_{ab}$$

Newtonian theory of *field* Love numbers



$M_{ab} = \lambda_2 \mathcal{E}_{ab}$



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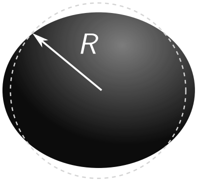
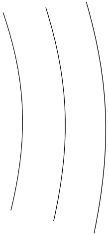
Newtonian theory of *field* Love numbers

M



$$M_{ab} = \lambda_2 \mathcal{E}_{ab}$$
$$= -\tfrac{2}{3} k_2 R^5 \mathcal{E}_{ab}$$



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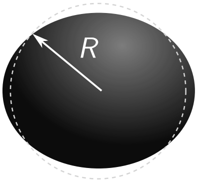

$$
\begin{aligned}
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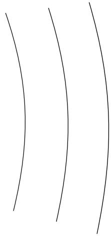
&= -\tfrac{2}{3} k_2 R^5 \mathcal{E}_{ab}
\end{aligned}
$$



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$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} \left[1 + 2k_2 \left(\frac{R}{r} \right)^5 \right]$$

Newtonian theory of *field* Love numbers

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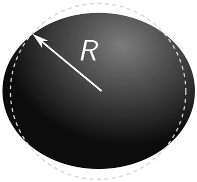

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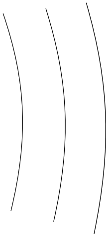
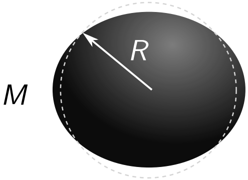


$$\mathcal{E}_{ab} = -\partial_{\langle a}\partial_{b\rangle} U_{\text{ext}}(\mathbf{0})$$

$$U = \frac{M}{r} - \sum_{l \geqslant 2} \frac{(l-2)!}{l!}\, x^L \mathcal{E}_L \left[1 + 2 k_l \left(\frac{R}{r}\right)^{2l+1}\right]$$

Tidal Love numbers k_l encode the body's **internal structure**

Newtonian theory of *field* Love numbers



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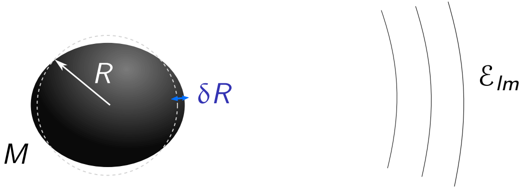
$$U = \frac{M}{r} - \sum_{l \geqslant 2} \sum_{|m| \leqslant l} \frac{(l-2)!}{l!}\, r^l \mathcal{E}_{lm} \left[1 + 2 k_l \left(\frac{R}{r} \right)^{2l+1} \right] Y_{lm}$$

Tidal Love numbers k_l encode the body's **internal structure**

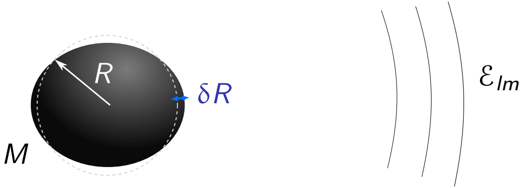
Newtonian theory of *shape* Love numbers



- The perturbation δR of the body's equilibrium **surface radius** $r = R$ is used to define the shape Love numbers h_l as:

$$\delta R(t, \theta, \phi) = -\sum_{lm} \frac{h_l}{l(l-1)} \frac{R^{l+2}}{M} \mathcal{E}_{lm}(t)\, Y_{lm}(\theta, \phi)$$

Newtonian theory of *shape* Love numbers



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- The equilibrium surface radius is an **equipotential** for a fluid body, implying

$$h_l = 1 + 2k_l$$

1. Motivations

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Relativistic theory of field Love numbers

- **Tidal** vs **response** decomposition:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \underbrace{h^{\text{tidal}}_{\alpha\beta}}_{\sim r^{l}} + \underbrace{h^{\text{resp}}_{\alpha\beta}}_{\sim r^{-(l+1)}}$$

[Binnington & Poisson, PRD **80** (2009) 084018 | Damour & Nagar, PRD **80** (2009) 084035]

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- Response induces shift in **field multipoles**:

$$\delta M_{lm} = \lambda^{\text{elec}}_{l}\, \mathcal{E}_{lm} \qquad \text{and} \qquad \delta S_{lm} = \lambda^{\text{mag}}_{l}\, \mathcal{B}_{lm}$$

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- *Nonspinning* BHs have **vanishing** static Love numbers

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- *Nonspinning* BHs have **vanishing** static Love numbers
- Effects of **spin**:
  - coupling between multipoles & m-dependence
  - mixing between electric and magnetic sectors

$\hookrightarrow$ **anisotropic** and **mode-coupled** response

[Binnington & Poisson, PRD **80** (2009) 084018 | Damour & Nagar, PRD **80** (2009) 084035]

Field Love numbers of a Kerr black hole

- Love numbers computed to **linear order** in spin $\chi \equiv S/M^2$

[Le Tiec & Casals, PRL **126** (2021) 131102 | Le Tiec, Casals & Franzin, PRD **103** (2021) 084021]

Field Love numbers of a Kerr black hole

- Love numbers computed to **linear order** in spin $\chi \equiv S/M^2$
- The modes of the mass/current **quadrupole moments** are

$$\delta M_{2m} \doteq \frac{im\chi}{180}\,(2M)^5\,\mathcal{E}_{2m} \quad \text{and} \quad \delta S_{2m} \doteq \frac{im\chi}{180}\,(2M)^5\,\mathcal{B}_{2m}$$

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- The associated **tidal Love numbers** are

$$\boxed{k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} \doteq -\frac{im\chi}{120}} \quad \text{and} \quad \boxed{k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} \doteq 0}$$

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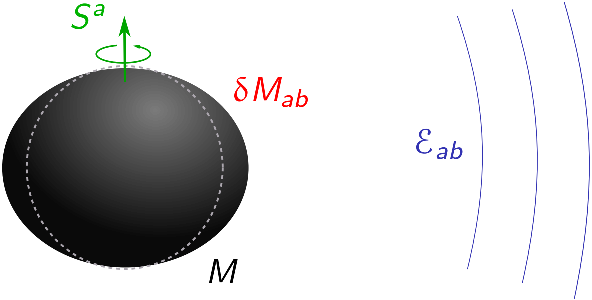
- For a black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \sim 10^{-3} \quad \longrightarrow \quad \textbf{extremely small response}$$

[Le Tiec & Casals, PRL **126** (2021) 131102 | Le Tiec, Casals & Franzin, PRD **103** (2021) 084021]

Geometric interpretation



$$\delta M_{ab} \propto M^3 \, S^c \, \mathcal{E}^d{}_{\langle a} \, \varepsilon_{b\rangle cd} \quad \longrightarrow \quad \text{tidal bulge } \textbf{rotated by } 45^\circ$$

[Le Tiec & Casals, PRL **126** (2021) 131102 | Le Tiec, Casals & Franzin, PRD **103** (2021) 084021]

Relativistic theory of shape Love numbers

- Define the shape Love numbers h_l from the perturbation $\delta\mathcal{R}$ of the **curvature radius** $\mathcal{R}$ at the body's surface as:

$$\delta\mathcal{R}(t,\theta,\phi) = -2\sum_{lm} \frac{l+2}{l}\, h_l\, \frac{R^{l-1}}{M}\, \mathcal{E}_{lm}(t)\, Y_{lm}(\theta,\phi)$$

[Damour & Lecian, PRD **80** (2009) 044017 | Landry & Poisson, PRD **89** (2014) 124011]

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- The Newtonian formula $h_l = 1 + 2k_l$ generalizes into

$$h_l = F_l(M/R) + 2G_l(M/R)\, k_l^{\mathrm{elec}}$$

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- In the **black hole limit** where $R \to 2M$, $k_l^{\mathrm{elec}} = 0$ and

$$h_l = \frac{l+1}{2(l-1)}\frac{(l!)^2}{(2l)!}$$

[Damour & Lecian, PRD **80** (2009) 044017 | Landry & Poisson, PRD **89** (2014) 124011]

Shape Love numbers of a Kerr black hole

Problem

The previous definition relies on the **spherical symmetry** of the background spacetime, and thus cannot be applied to Kerr BHs

[Ribes Metidieri *et al.*, PRD **110** (2024) 024069 | Le Tiec *et al.*, in preparation (2026)]

Shape Love numbers of a Kerr black hole

Problem

The previous definition relies on the **spherical symmetry** of the background spacetime, and thus cannot be applied to Kerr BHs

Proposal

Define shape Love numbers from a geometric, quasi-local notion of **horizon multipole moments** I_{lm} and L_{lm}, according to

$$\delta I_{lm} = \sum_{l'm'} \left(\lambda^{I\mathcal{E}}_{lm,l'm'} \, \mathcal{E}_{l'm'} + \lambda^{I\mathcal{B}}_{lm,l'm'} \, \mathcal{B}_{l'm'} \right)$$

$$\delta L_{lm} = \sum_{l'm'} \left(\lambda^{L\mathcal{E}}_{lm,l'm'} \, \mathcal{E}_{l'm'} + \lambda^{L\mathcal{B}}_{lm,l'm'} \, \mathcal{B}_{l'm'} \right)$$

[Ribes Metidieri *et al.*, PRD **110** (2024) 024069 | Le Tiec *et al.*, in preparation (2026)]

Horizon multipole moments of a NEH

Definition

The **shape** and **current** multipole moments I_{lm} and L_{lm} of a non-expanding horizon (NEH) are defined by

$$I_{lm} + \mathrm{i} L_{lm} \equiv -\oint_{\mathcal{S}} \Psi_2 \, \mathring{Y}_{lm} \, \mathrm{d}S$$

[Ashtekar *et al.*, CQG **21** (2004) 2549 | Ashtekar *et al.*, JHEP **2022** (2022) 28]

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Properties

- The Coulomb-type curvature information is encoded into the **Weyl scalar** Ψ_2

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Properties

- The Coulomb-type curvature information is encoded into the **Weyl scalar** Ψ_2
- This definition is **independent** of the choice of $\mathcal{S}$

[Ashtekar *et al.*, CQG **21** (2004) 2549 | Ashtekar *et al.*, JHEP **2022** (2022) 28]

Horizon multipoles of a *perturbed* NEH

- For a closed surface $\mathcal{S}$ embedded in spacetime, the invariant information is encoded into the **complex curvature**

$$\mathcal{K} \equiv -\frac{1}{4}\underset{\substack{\uparrow \\ \text{intrinsic}}}{\mathcal{R}} + \mathrm{i}\underset{\substack{\uparrow \\ \text{extrinsic}}}{\mathcal{X}} \overset{\mathcal{S}}{=} \Psi_2 + \rho\rho' - \sigma\sigma'$$

[Le Tiec, Gourgoulhon & Casals, in preparation (2026)]

Horizon multipoles of a *perturbed* NEH

- For a closed surface $\mathcal{S}$ embedded in spacetime, the invariant information is encoded into the **complex curvature**

$$\mathcal{K} \equiv -\frac{1}{4}\underset{\substack{\uparrow \\ \text{intrinsic}}}{\mathcal{R}} + \mathrm{i}\underset{\substack{\uparrow \\ \text{extrinsic}}}{\mathcal{X}} \overset{\mathcal{S}}{=} \Psi_2 + \rho\rho' - \sigma\sigma'$$

- Accordingly, the **shape** and **current** multipole moments I_{lm} and L_{lm} of $\mathcal{S}$ are defined by

$$I_{lm} + \mathrm{i}L_{lm} \equiv -\oint_{\mathcal{S}} \mathcal{K}\, \mathring{Y}_{lm}\, \mathrm{d}S$$

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- The linear perturbations δI_{lm} and δL_{lm} are **gauge invariant** $\hookrightarrow$ geometrical definition of black hole shape Love numbers

[Le Tiec, Gourgoulhon & Casals, in preparation (2026)]

Take-home messages

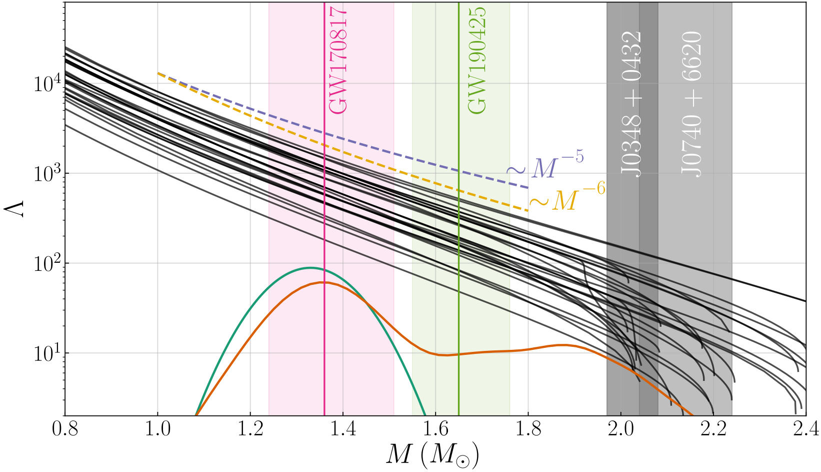
1. Black hole *field* Love numbers **do not vanish** in general
2. Kerr black holes have a **purely dissipative** (no conservative) tidal response
3. Horizon multipoles provide a **geometric characterization** of black hole *shape* Love numbers
4. Gravitational-wave observations enable **precision tests** of:
   - the no-hair theorem (stationary geometry)
   - tidal deformability (dynamical response)

From geometry to observable signatures

Additional Material

First constraints from GW observations

$\Lambda \equiv \lambda_2/M^5 \propto k_2/C^5$



[Chatzioannou, GRG **52** (2020) 109]

First constraints from GW observations

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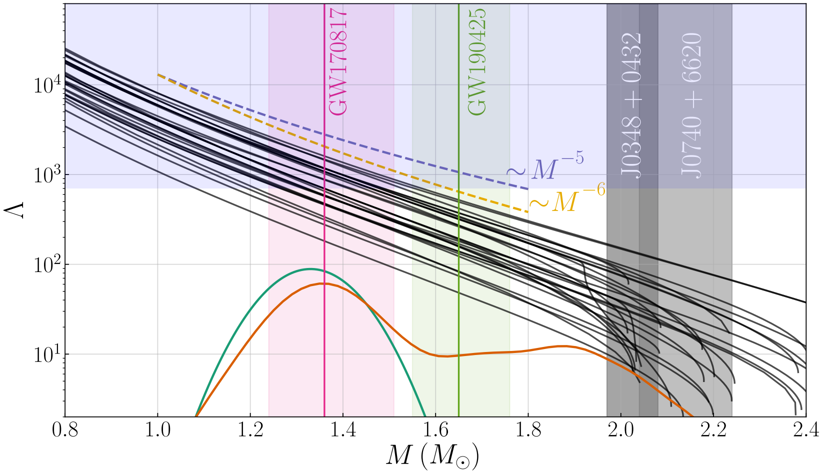
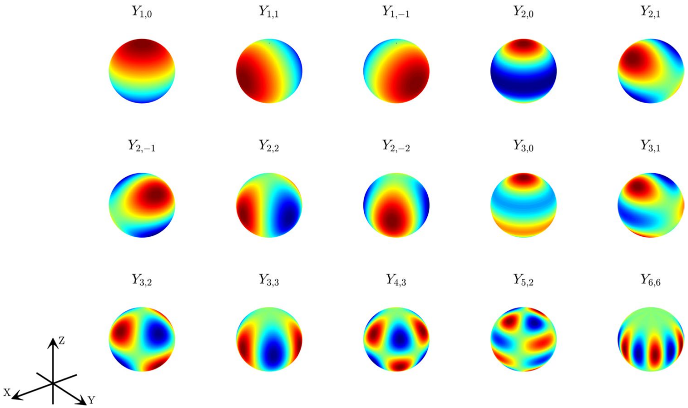


[Chatziioannou, GRG **52** (2020) 109]

Field multipoles in Newtonian gravity



Angular structure encoded into spherical harmonics $Y_{lm}(\theta, \phi)$

Geroch-Hansen multipole moments

- Consider a **stationary** spacetime with **timelike** Killing field ξ^a
- Define the norm λ and twist ω_a of ξ^a according to

$$\lambda \equiv -\xi^a \xi_a \qquad \text{and} \qquad \omega_a \equiv \varepsilon_{abcd} \xi^b \nabla^c \xi^d$$

- In **vacuum**, one can shown that

$$\nabla_{[a} \omega_{b]} = 0, \quad \text{implying} \quad \omega_a = \nabla_a \omega$$

- Define a **complex potential** $\Phi = \Phi_M + \mathrm{i}\Phi_S$, with

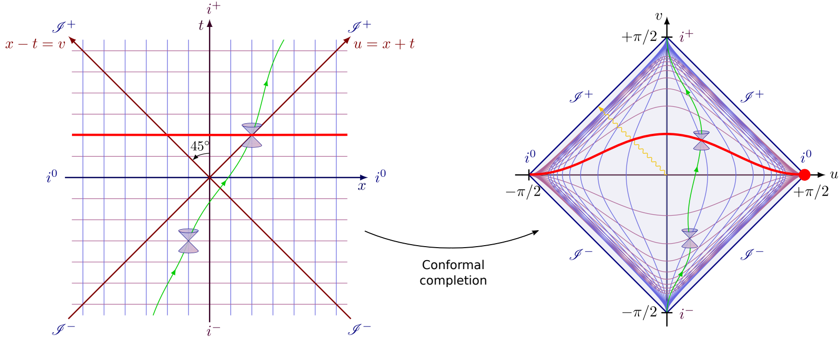
$$\Phi_M \equiv \frac{1 - \lambda^2 - \omega^2}{4\lambda} \qquad \text{and} \qquad \Phi_S \equiv \frac{\omega}{2\lambda}$$

$\hookrightarrow$ relativistic generalization of the Newtonian potential U

[Geroch, JMP **11** (1970) 2580 | Hansen, JMP **15** (1974) 46]

Geroch-Hansen multipole moments

Idea: define **mass-type** and **current-type** multipoles M_{lm} and S_{lm} from the potential Φ at **spatial infinity** i^0



Physical spacetime **Unphysical spacetime**

[Geroch, JMP **11** (1970) 2580 | Hansen, JMP **15** (1974) 46 | tikz.net/relativity_penrose_diagram]

Field multipoles of a Kerr black hole

- For a Kerr black hole of mass M and spin $S = Ma = M^2\chi$, the norm and twist read (using BL coordinates)

$$\lambda = \frac{r^2 - 2Mr + a^2\cos^2\theta}{r^2 + a^2\cos^2\theta} \qquad \text{and} \qquad \omega = \frac{2Ma\cos\theta}{r^2 + a^2\cos^2\theta}$$

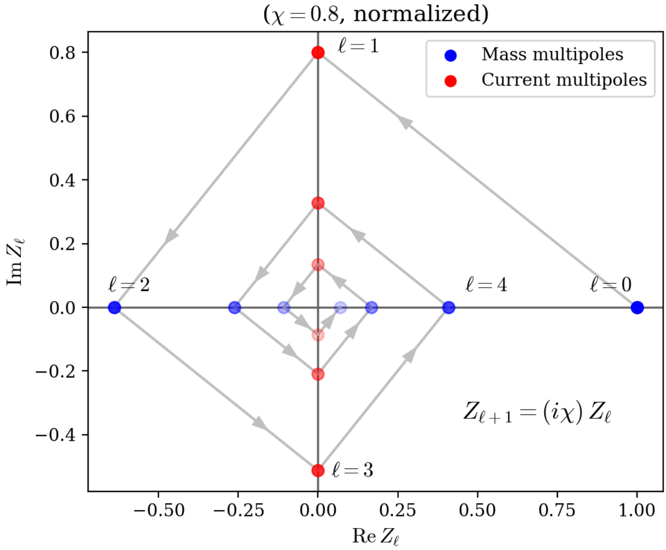
- Due to the **axisymmetry** of the Kerr metric, the non-zero multipoles all have $m = 0$. They read

$$\boxed{M_{l0} + \mathrm{i}S_{l0} = M(\mathrm{i}a)^l} \quad \Longleftrightarrow \quad Z_l = (\mathrm{i}\chi)^l$$

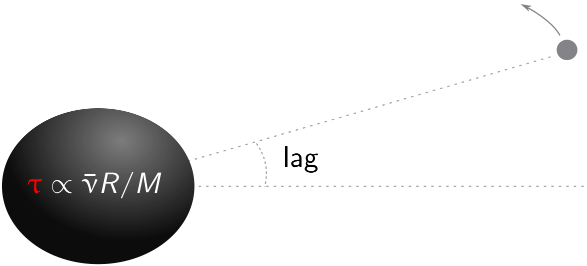
Remark: the Kerr black hole is not merely **oblate**; it is also infinitely structured, yet completely **rigid**

[Hansen, JMP **15** (1974) 46]

Field multipoles of a Kerr black hole

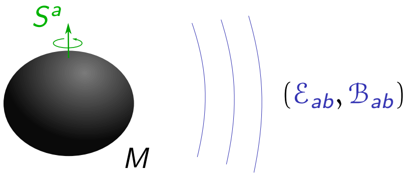


Tidal dissipation: lag, heating and torquing



$$M_{ab}(t) = -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t) - \tau \dot{\mathcal{E}}_{ab}(t) + \cdots \right]$$
$$= -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t - \tau) + \cdots \right]$$

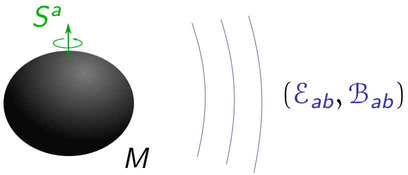
Tidal torquing



- Any spinning body interacting with a tidal environment suffers a **tidal torquing** [Thorne & Hartle, PRD 1985]

$$\langle \dot{S}^a \rangle = -\varepsilon^{abc} \langle M_{bd} \mathcal{E}^d{}_c + S_{bd} \mathcal{B}^d{}_c \rangle$$

Tidal torquing



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$$\langle \dot{S}^a \rangle = -\varepsilon^{abc} \langle M_{bd} \mathcal{E}^d{}_c + S_{bd} \mathcal{B}^d{}_c \rangle$$

- Applied to a Kerr black hole this yields

$$\langle \dot{S} \rangle \propto M^3 S \left[\langle \mathcal{E}^2 \rangle + \langle \mathcal{B}^2 \rangle \right]$$

$\hookrightarrow$ **full agreement** with analysis of [Poisson, PRD 2004]

A burst of activity on BH tidal deformability

- Other **backgrounds**, generic **spin-s** fields and higher **dimensions**
  [Hui *et al.*, JCAP 2021; Pereñiguez & Cardoso, PRD 2022; Rodriguez *et al.*, PRD 2023; Charalambous & Ivanov, JHEP 2023; Charalambous, JHEP 2024]

- **Dissipative nature** of Kerr BH tidal deformability
  [Chia, PRD 2021; Goldberger *et al.*, JHEP 2021; Charalambous, JHEP 2021; Prasad Bhatt *et al.*, PRD 2023]

- **Hidden symmetry** and vanishing BH field Love numbers
  [Charalambous *et al.*, PRL 2021; Hui *et al.*, JCAP 2022; Charalambous *et al.*, JHEP 2022; Achour *et al.*, JHEP 2022; Hui *et al.*, JHEP 2022; Berens *et al.*, JCAP 2023; Katagiri *et al.*, PRD 2023; Rai & Santoni 2024]

- **Scattering amplitudes** and vanishing BH field Love numbers
  [Creci *et al.*, PRD 2021; Ivanov & Zhou, PRL 2023; Saketh *et al.*, PRD 2024]

- Effective Field Theory, matching and **logarithmic corrections**
  [Ivanov & Zhou, PRD 2023]

- **Nonlinearities** in the BH field Love numbers
  [De Luca *et al.*, PRD 2023; Maria Riva *et al.* 2023; Hadad *et al.* 2024; Combaluzier-Szteinsznaider *et al.* 2024; Kehagias & Riotto 2024]

A burst of activity on BH tidal deformability

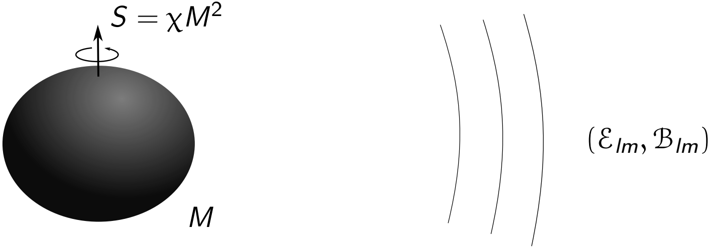
Two recent **review articles**:

- *Love numbers of black holes and compact objects*
  Rodríguez, Santoni & Solomon
  arXiv:gr-qc/2604.08653
- *Tidal response of compact objects*
  Chakraborty & Pani
  arXiv:gr-qc/2604.08679

Do BHs have *zero* field Love numbers?

Reference	**Background**	**Tidal field**
[Binnington & Poisson 2009]	Schwarzschild	weak, generic l
[Damour & Nagar 2009]	Schwarzschild	weak, generic l
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $l = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(S^2)$	weak, $(l, m) = (2, 0)$
[Le Tiec & Casals 2021]	Exact Kerr	weak, generic l
[Chia 2021]	Exact Kerr	weak, generic l
[Goldberger et al. 2021]	Exact Kerr	weak, generic l
[Charalambous et al. 2021]	Exact Kerr	weak, generic l
[Hui 2021]	Schwar-(A)dS	weak, generic l

Investigating Kerr's Love



$$(\mathcal{E}_{lm}, \mathcal{B}_{lm}) \to \psi_0 \to \Psi \to h_{\alpha\beta} \to (M_{lm}, S_{lm}) \to \lambda_{lm}^{M/S, \mathcal{E}/\mathcal{B}}$$

Metric reconstruction through the Hertz potential Ψ

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c\to\infty} \psi_0^{lm} \propto \mathcal{E}_{lm}\, r^{l-2} \left[1 + 2k_{lm}\,(R/r)^{2l+1}\right] {}_2Y_{lm}(\theta,\phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{lm} \propto \left[\mathcal{E}_{lm} + \tfrac{3i}{l+1}\,\mathcal{B}_{lm}\right] R_{lm}(r)\; {}_2Y_{lm}(\theta,\phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{lm}(r) = \underbrace{r^{l-2}\,(1+\cdots)}_{\text{tidal field } R_{lm}^{\text{tidal}}} + k_{lm}\,\underbrace{r^{-l-3}\,(1+\cdots)}_{\text{linear response } R_{lm}^{\text{resp}}}$$

Why analytic continuation?

$$R_{lm}(r) = \underbrace{r^{l-2}\,(1+\cdots)}_{\text{tidal field } R_{lm}^{\text{tidal}}} + k_{lm}\,\underbrace{r^{-l-3}\,(1+\cdots)}_{\text{linear response } R_{lm}^{\text{resp}}}$$

Ambiguity in the linear response [Fang & Lovelace 2005; Gralla 2018]

The decaying solution R_{lm}^{resp} is affected by a radial coord. transfo.

Ambiguity in the tidal field [Pani, Gualtieri, Maselli & Ferrari 2015]

The growing solution $R_{lm}^{\text{tidal}} + \alpha R_{lm}^{\text{resp}}$ still qualifies as a tidal solution

Kerr black hole linear response

$$R_{lm}(r) = \underbrace{R^{\text{tidal}}_{lm}(r)}_{\sim r^{l-2}} + 2k_{lm} \underbrace{R^{\text{resp}}_{lm}(r)}_{\sim r^{-(l+3)}}$$

- The coefficients k_{lm} can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$$k_{lm} = -im\chi \frac{(l+2)!(l-2)!}{4(2l+1)!(2l)!} \prod_{n=1}^{l} \left[n^2(1-\chi^2) + m^2\chi^2 \right]$$

- The linear response vanishes identically when:
  - the black hole spin vanishes ($\chi = 0$)
  - the tidal field is axisymmetric ($m = 0$)
- Reconstruct the Kerr black hole response $h^{\text{resp}}_{\alpha\beta}$ via Ψ^{resp}

Tidal Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\mathrm{elec}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\mathrm{mag}} \mathcal{B}_{ab}$$

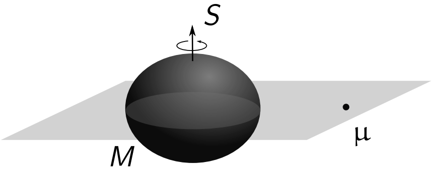
- For a spinning black hole we have the more general tensorial relations

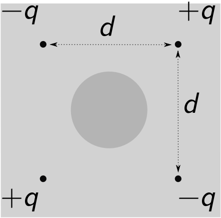
$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}^{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}^{cd}$$

- To *linear order* in the black hole spin vector S^a we find

$$\delta M_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{E}^d{}_{\langle a} \varepsilon_{b\rangle cd}$$
$$\delta S_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{B}^d{}_{\langle a} \varepsilon_{b\rangle cd}$$

Newtonian static quadrupolar tide





$$\mathcal{E}_{ab} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

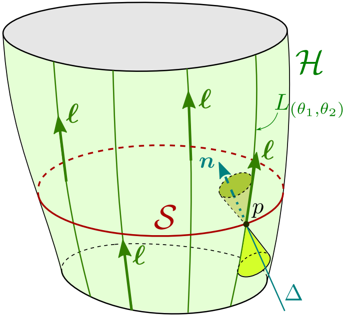
$$\delta M_{ab} \doteq \underset{\substack{\uparrow \\ \frac{\chi}{180}(2M)^5\frac{\mu}{r^3} = qd^2}}{3Q} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Non-expanding horizons (NEHs)

Definition

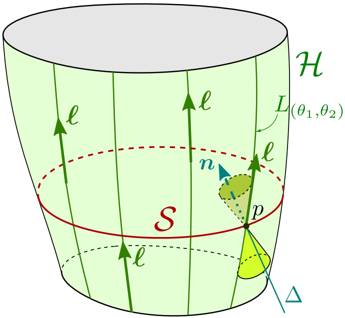
A **null hypersurface** $\mathcal{H}$ is a non-expanding horizon iif:

- $\mathcal{H} \sim \mathbb{R} \times \mathbb{S}^2$
- $\forall \boldsymbol{\ell} : \theta_{(\boldsymbol{\ell})} = 0$



[Ashtekar, Fairhurst & Krishnan, PRD **62** (2000) 104025]

Non-expanding horizons (NEHs)



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Consequences

- Raychaudhuri equation implies **vanishing shear** σ_{ab} of $\boldsymbol{\ell}$
- Induced geometry on $\mathcal{H}$ is **invariant** along $\boldsymbol{\ell}$

[Ashtekar, Fairhurst & Krishnan, PRD **62** (2000) 104025]

Non-expanding horizons (NEHs)

Main properties

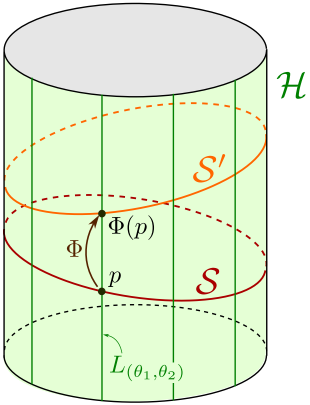
- All cross-sections are **isometric**
  $\hookrightarrow$ well-defined surface area

$$A \equiv \oint_{\mathcal{S}} \boldsymbol{\varepsilon}$$

- **Rotation 1-form** ω_a defined by

$$\mathcal{D}_a \ell^b \stackrel{\mathcal{H}}{=} \omega_a \ell^b,$$

with $\mathcal{D}_a$ the **intrinsic connection** induced on $\mathcal{H}$



[Ashtekar, Beetle & Lewandowski, CQG **19** (2002) 1195]

Coulomb-type Weyl curvature scalar

- Given a complex null tetrad $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \boldsymbol{m}^*)$, the Coulomb-type curvature information is encoded into the **Weyl scalar**

$$\Psi_2 \equiv C_{abcd}\ell^a m^b m^{*c} n^d$$

- On a NEH $\mathcal{H}$, this curvature scalar is **frame-invariant**, with

$$\begin{aligned} \operatorname{Re}\Psi_2 &\overset{\mathcal{H}}{=} -\tfrac{1}{4}\,\mathcal{R} \quad &\longrightarrow \quad &\textbf{shape} \text{ (intrinsic curvature)} \\ (\operatorname{Im}\Psi_2)\,\varepsilon_{ab} &\overset{\mathcal{H}}{=} \mathcal{D}_{[a}\omega_{b]} \quad &\longrightarrow \quad &\textbf{rotational structure} \text{ (vorticity)} \end{aligned}$$

$\hookrightarrow$ natural candidate for horizon multipoles on $\mathcal{S}$

[Ashtekar, Beetle & Lewandowski, CQG **19** (2002) 1195]

Effective sources for horizon multipoles

	Electrostatics	Horizon (shape)
Volume	$Q_{lm} = \int_{\mathbb{R}^3} r^l \rho \, Y_{lm} \, \mathrm{d}V$	
Surface	$Q_{lm} \propto \oint_{\mathbb{S}^2} \tilde{\rho} \, Y_{lm} \, \mathrm{d}S$	$I_{lm} \propto \oint_{\mathcal{S}} \mathcal{R} \, Y_{lm} \, \mathrm{d}S$
Source	Surface charge density $\tilde{\rho}$	Intrinsic curvature $\mathcal{R}$

thin shell

	Magnetostatics	Horizon (current)
Volume	$\mu_{lm} = \int_{\mathbb{R}^3} r^l \varepsilon^{AB} j_B \, D_A Y_{lm} \, \mathrm{d}V$	
Surface	$\mu_{lm} \propto \oint_{\mathbb{S}^2} \varepsilon^{AB} D_A \tilde{j}_B \, Y_{lm} \, \mathrm{d}S$	$L_{lm} \propto \oint_{\mathcal{S}} \varepsilon^{AB} D_A \omega_B \, Y_{lm} \, \mathrm{d}S$
Source	Curl of surface current $\tilde{j}_A$	Curl of rotation 1-form ω_A

Horizon multipoles of a Kerr black hole

- Due to the **axisymmetry** of the Kerr metric, all non-zero multipoles have $m = 0$. The **reflection symmetry** accross the equatorial plane further implies

$$\forall n \in \mathbb{N}, \quad I_{2n+1,0} = 0 \quad \text{and} \quad L_{2n,0} = 0$$

[Gourgoulhon, Le Tiec & Casals, to appear in PRD (2026), gr-qc/2602.05823]

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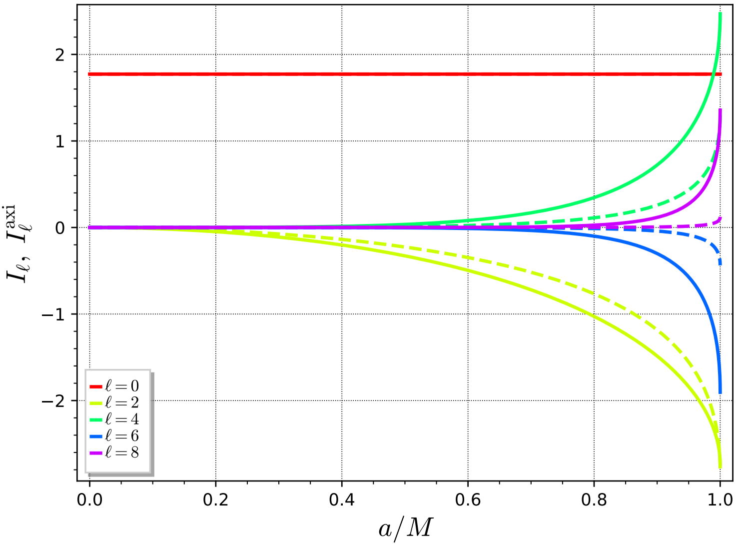
- Moreover, in the regime of **small-spin** values $0 < \chi \ll 1$, the non-zero horizon multipoles behave as

$$\boxed{I_{l0} + \mathrm{i} L_{l0} \sim \alpha_l \, (\mathrm{i}\chi)^l}$$

$\hookrightarrow$ **same scaling** as normalized field multipoles $Z_l = (\mathrm{i}\chi)^l$

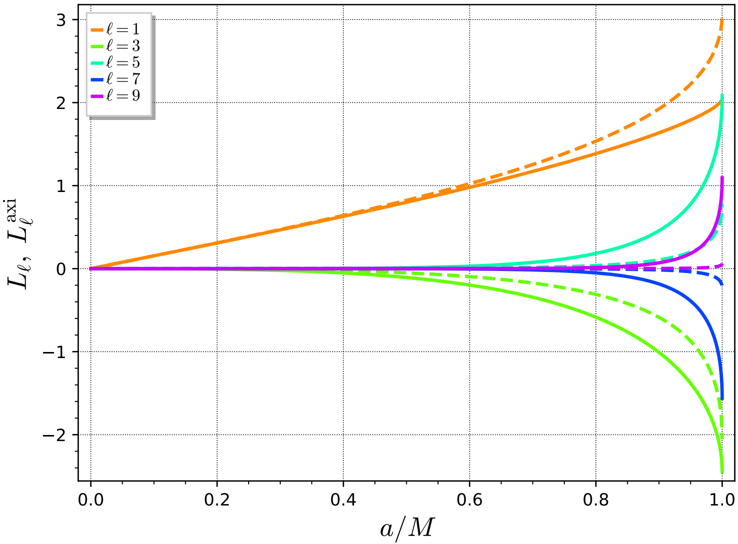
[Gourgoulhon, Le Tiec & Casals, to appear in PRD (2026), gr-qc/2602.05823]

Shape multipoles of a Kerr black hole



[Gourgoulhon, Le Tiec & Casals, to appear in PRD (2026), gr-qc/2602.05823]

Current multipoles of a Kerr black hole



[Gourgoulhon, Le Tiec & Casals, to appear in PRD (2026), gr-qc/2602.05823]