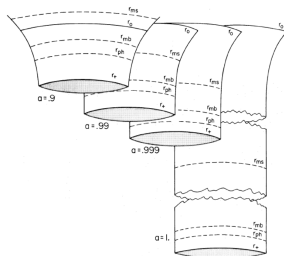
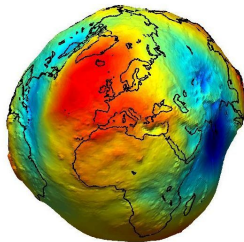


Tidal deformations of black holes

Alexandre Le Tiec

LUX, Observatoire de Paris | CNRS
Instituto de Física Teórica | UNESP



TIDES

Low tide

High tide



High tide



Moon

Low tide

① Motivations

② Newtonian tides

③ Black hole field Love numbers

④ Black hole shape Love numbers

① Motivations

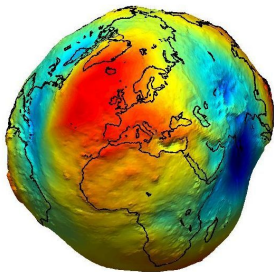
② Newtonian tides

③ Black hole field Love numbers

④ Black hole shape Love numbers

Do isolated black holes have hair?

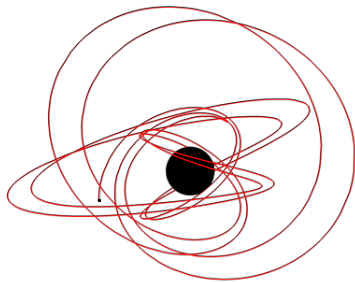
Geodesy



M_{lm} arbitrary

rich internal structure

Black holes



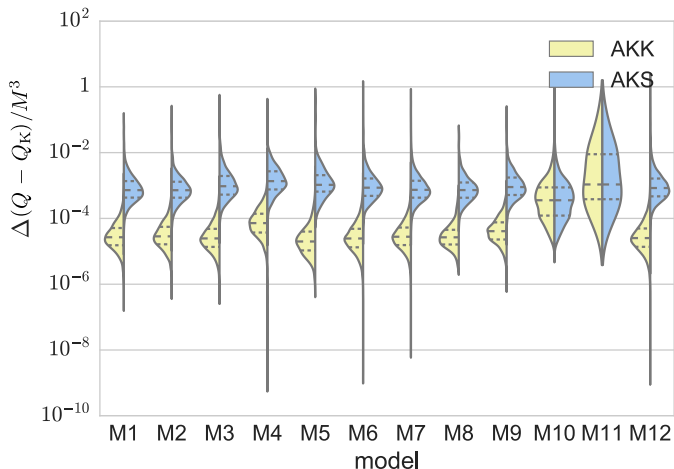
$$M_{l0} + iS_{l0} = M(ia)^l$$

completely fixed by (M, a)

Objective: measure the *multipolar structure* of astrophysical BHs

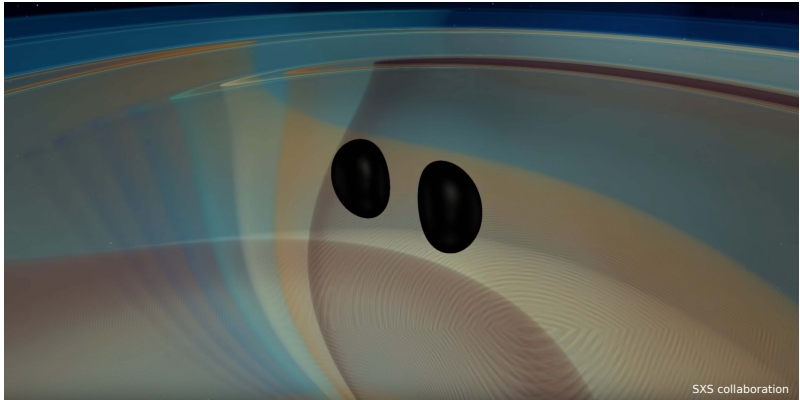
LISA constraints from EMRIs

Waveform $\rightarrow$ measure $(M_{lm}, S_{lm}) \rightarrow$ test $Z_l = (i\chi)^l$



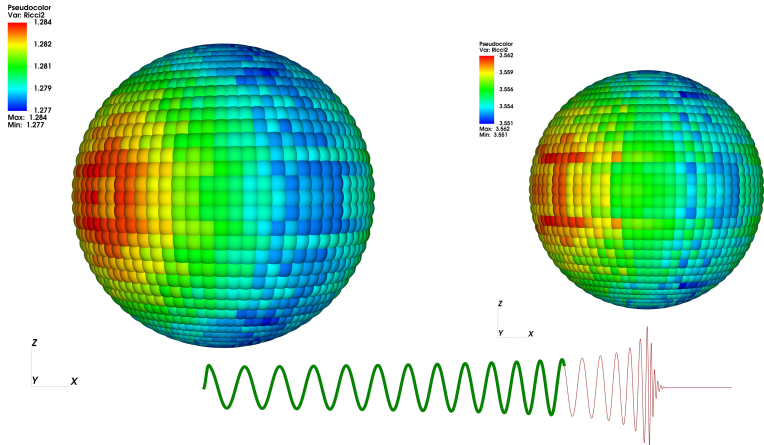
Do tidally-interacting black holes deform?

Isolated BH: **no hair** $\longrightarrow$ Binary BH: **do they respond?**



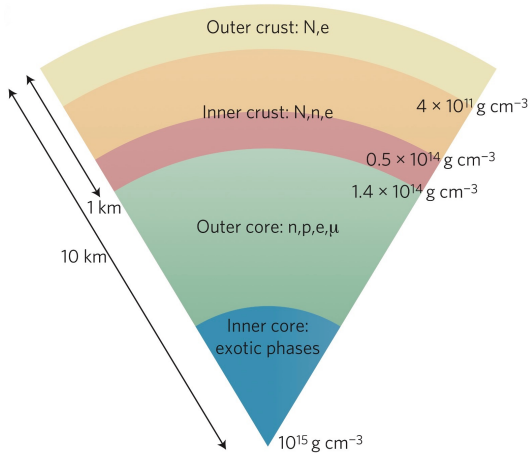
Objective: measure the *dynamical response* of astrophysical BHs

Black hole tomography by GW observations



Waveform $\rightarrow$ measure $(I_{lm}, L_{lm}) \rightarrow$ shape Love numbers

What is the internal structure of a NS?



GW observations as probes of **neutron star equation of state**

① Motivations

② Newtonian tides

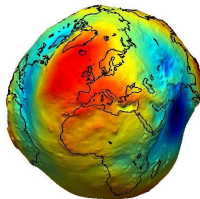
③ Black hole field Love numbers

④ Black hole shape Love numbers

Field multipoles in Newtonian gravity

- Gravitational field:

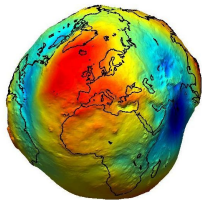
$$U(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'$$



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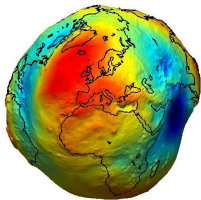
- Multipole expansion:

$$U(r, \theta, \phi) = \sum_{l \geq 0} \sum_{|m| \leq l} \frac{4\pi}{2l+1} \frac{M_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi)$$

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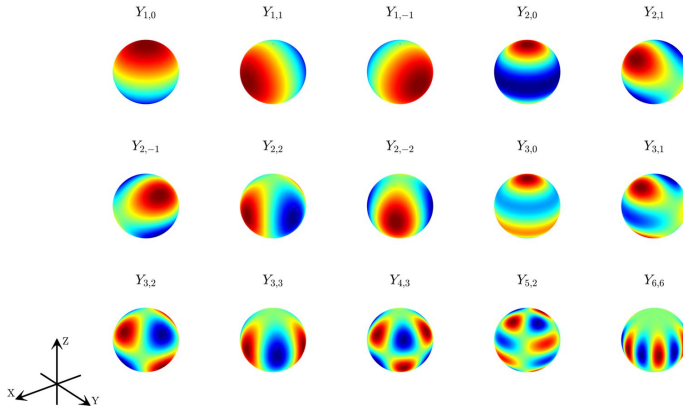
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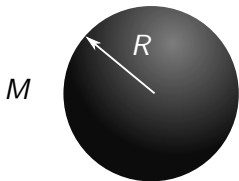
$$M_{lm} = \int_{\mathbb{R}^3} \rho(\mathbf{x}) r^l Y_{lm}^* d^3\mathbf{x}$$

Field multipoles in Newtonian gravity



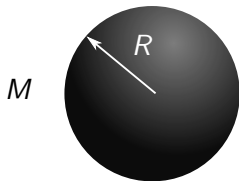
Angular structure encoded into spherical harmonics $Y_{lm}(\theta, \phi)$

Newtonian theory of *field* Love numbers



$$U = \frac{M}{r}$$

Newtonian theory of *field* Love numbers

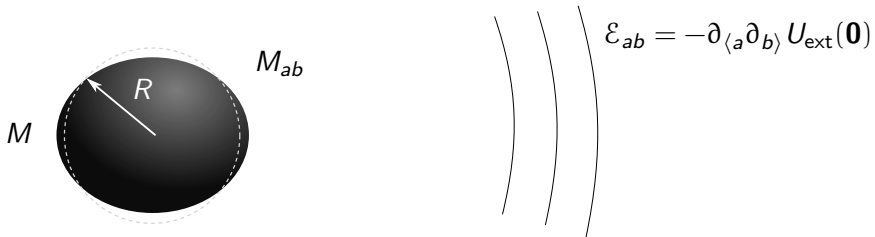


Three vertical, slightly curved lines representing a tidal field. To their right is the equation $\mathcal{E}_{ab} = -\partial_{\langle a} \partial_{b \rangle} U_{\text{ext}}(\mathbf{0})$.

$$\mathcal{E}_{ab} = -\partial_{\langle a} \partial_{b \rangle} U_{\text{ext}}(\mathbf{0})$$

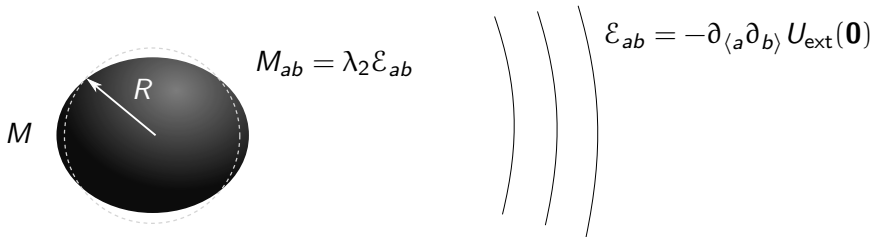
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab}$$

Newtonian theory of *field* Love numbers



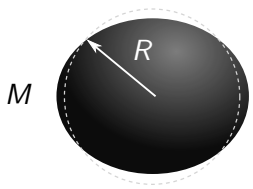
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} + \frac{3}{2} \frac{x^a x^b}{r^5} M_{ab}$$

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Newtonian theory of *field* Love numbers



A diagram of a sphere of mass M and radius R . The sphere is shown with a solid black surface and a dashed white outline representing its undeformed state. An arrow labeled R points from the center to the surface. To the left of the sphere is the label M .

$$M_{ab} = \lambda_2 \mathcal{E}_{ab} = -\frac{2}{3} k_2 R^5 \mathcal{E}_{ab}$$

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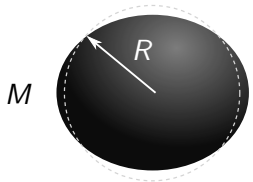


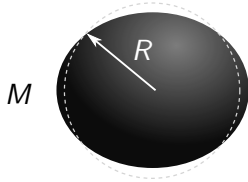
Diagram illustrating a sphere of mass M and radius R . The sphere is shown with a dashed outline representing a perturbed state.

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Newtonian theory of *field* Love numbers



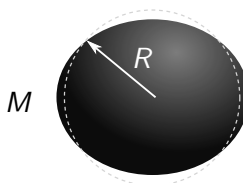
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Tidal Love numbers k_l encode the body's **internal structure**

Newtonian theory of *field* Love numbers



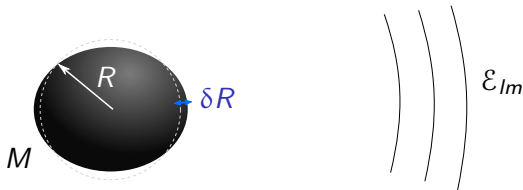
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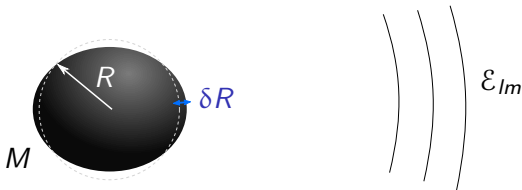
Newtonian theory of *shape* Love numbers



- The perturbation δR of the body's equilibrium **surface radius** $r = R$ is used to define the shape Love numbers h_l as:

$$\delta R(t, \theta, \phi) = - \sum_{lm} \frac{h_l}{l(l-1)} \frac{R^{l+2}}{M} \epsilon_{lm}(t) Y_{lm}(\theta, \phi)$$

Newtonian theory of *shape* Love numbers



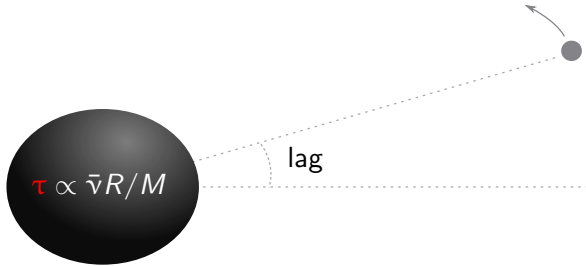
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- The equilibrium surface radius is an **equipotential** for a fluid body, implying

$$h_l = 1 + 2k_l$$

Tidal dissipation: lag, heating and torquing



$$\begin{aligned} M_{ab}(t) &= -\frac{2}{3} k_2 R^5 [\mathcal{E}_{ab}(t) - \tau \dot{\mathcal{E}}_{ab}(t) + \cdots] \\ &= -\frac{2}{3} k_2 R^5 [\mathcal{E}_{ab}(t - \tau) + \cdots] \end{aligned}$$

① Motivations

② Newtonian tides

③ Black hole field Love numbers

④ Black hole shape Love numbers

Relativistic theory of field Love numbers

- **Tidal** vs **response** decomposition:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \underbrace{h_{\alpha\beta}^{\text{tidal}}}_{\sim r^l} + \underbrace{h_{\alpha\beta}^{\text{resp}}}_{\sim r^{-(l+1)}}$$

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$$\delta M_{lm} = \lambda_l^{\text{elec}} \mathcal{E}_{lm} \quad \text{and} \quad \delta S_{lm} = \lambda_l^{\text{mag}} \mathcal{B}_{lm}$$

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- *Nonspinning* BHs have **vanishing** static Love numbers
 - Effects of **spin**:
 - coupling between multipoles & m -dependence
 - mixing between electric and magnetic sectors
- ↪ **anisotropic** and **mode-coupled** response

Field Love numbers of a Kerr black hole

- Love numbers computed to **linear order** in spin $\chi \equiv S/M^2$

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- The modes of the mass/current **quadrupole moments** are

$$\delta M_{2m} \doteq \frac{im\chi}{180} (2M)^5 \mathcal{E}_{2m} \quad \text{and} \quad \delta S_{2m} \doteq \frac{im\chi}{180} (2M)^5 \mathcal{B}_{2m}$$

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$$\boxed{k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} \doteq -\frac{im\chi}{120}} \quad \text{and} \quad \boxed{k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} \doteq 0}$$

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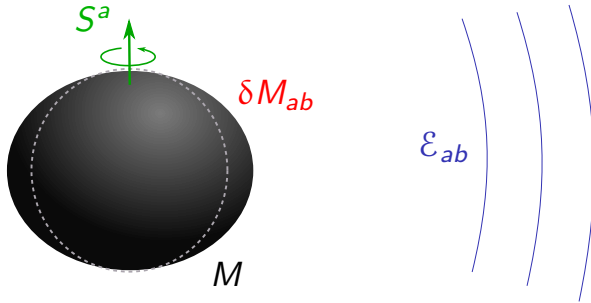
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- For a black hole spin $\chi = 0.1$ this gives

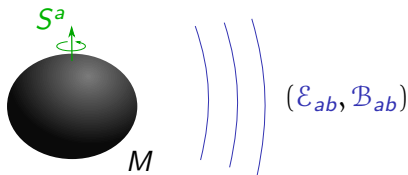
$$|k_{2,\pm 2}| \sim 10^{-3} \quad \longrightarrow \quad \textbf{extremely small response}$$

Geometric interpretation



$$\delta M_{ab} \propto M^3 S^c \epsilon^d_{\langle a} \epsilon_{b \rangle cd} \longrightarrow \text{tidal bulge rotated by } 45^\circ$$

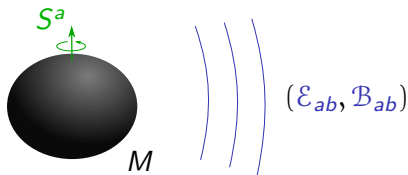
Tidal torquing



- Any spinning body interacting with a tidal environment suffers a **tidal torquing** [Thorne & Hartle, PRD 1985]

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$$\langle \dot{S}^a \rangle = -\epsilon^{abc} \langle M_{bd} \mathcal{E}^d_c + S_{bd} \mathcal{B}^d_c \rangle$$

- Applied to a Kerr black hole this yields

$$\langle \dot{S} \rangle \propto M^3 S [\langle \mathcal{E}^2 \rangle + \langle \mathcal{B}^2 \rangle]$$

↪ **full agreement** with analysis of [Poisson, PRD 2004]

A burst of activity on BH tidal deformability

- Other **backgrounds**, generic **spin- s** fields and higher **dimensions**
[Hui *et al.*, JCAP 2021; Pereñíguez & Cardoso, PRD 2022; Rodriguez *et al.*, PRD 2023; Charalambous & Ivanov, JHEP 2023; Charalambous, JHEP 2024]
- **Dissipative nature** of Kerr BH tidal deformability
[Chia, PRD 2021; Goldberger *et al.*, JHEP 2021; Charalambous, JHEP 2021; Prasad Bhatt *et al.*, PRD 2023]
- **Hidden symmetry** and vanishing BH field Love numbers
[Charalambous *et al.*, PRL 2021; Hui *et al.*, JCAP 2022; Charalambous *et al.*, JHEP 2022; Achour *et al.*, JHEP 2022; Hui *et al.*, JHEP 2022; Berens *et al.*, JCAP 2023; Katagiri *et al.*, PRD 2023; Rai & Santoni 2024]
- **Scattering amplitudes** and vanishing BH field Love numbers
[Creci *et al.*, PRD 2021; Ivanov & Zhou, PRL 2023; Saketh *et al.*, PRD 2024]
- Effective Field Theory, matching and **logarithmic corrections**
[Ivanov & Zhou, PRD 2023]
- **Nonlinearities** in the BH field Love numbers
[De Luca *et al.*, PRD 2023; Maria Riva *et al.* 2023; Hadad *et al.* 2024; Combaluzier-Szteinsznaider *et al.* 2024; Kehagias & Riotto 2024]

A burst of activity on BH tidal deformability

Two recent **review articles**:

- *Love numbers of black holes and compact objects*
Rodríguez, Santoni & Solomon
arXiv:gr-qc/2604.08653
- *Tidal response of compact objects*
Chakraborty & Pani
arXiv:gr-qc/2604.08679

① Motivations

② Newtonian tides

③ Black hole field Love numbers

④ Black hole shape Love numbers

For a nonspinning compact body

- Define the shape Love numbers h_l from the perturbation $\delta\mathcal{R}$ of the **curvature radius** $\mathcal{R}$ at the body's surface as:

$$\delta\mathcal{R}(t, \theta, \phi) = -2 \sum_{lm} \frac{l+2}{l} h_l \frac{R^{l-1}}{M} \varepsilon_{lm}(t) Y_{lm}(\theta, \phi)$$

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- In the **black hole limit** where $R \rightarrow 2M$, $k_l^{\text{elec}} = 0$ and

$$h_l = \frac{l+1}{2(l-1)} \frac{(l!)^2}{(2l)!}$$

For a spinning black hole

Problem

The previous definition relies on the **spherical symmetry** of the background spacetime, and thus cannot be applied to Kerr BHs

For a spinning black hole

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Proposal

Define shape Love numbers from a geometric, quasi-local notion of **horizon multipole moments** I_{lm} and L_{lm} , according to

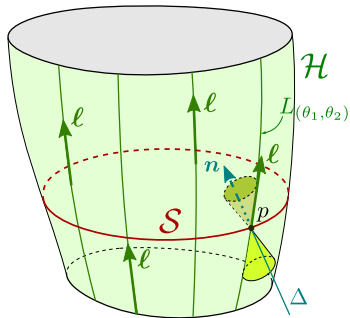
$$\delta I_{lm} = \sum_{l'm'} \left(\lambda_{lm,l'm'}^{I\mathcal{E}} \mathcal{E}_{l'm'} + \lambda_{lm,l'm'}^{I\mathcal{B}} \mathcal{B}_{l'm'} \right)$$
$$\delta L_{lm} = \sum_{l'm'} \left(\lambda_{lm,l'm'}^{L\mathcal{E}} \mathcal{E}_{l'm'} + \lambda_{lm,l'm'}^{L\mathcal{B}} \mathcal{B}_{l'm'} \right)$$

Non-expanding horizons (NEHs)

Definition

A **null hypersurface** $\mathcal{H}$ is a non-expanding horizon iif:

- $\mathcal{H} \sim \mathbb{R} \times \mathbb{S}^2$
- $\forall \ell : \theta_{(\ell)} = 0$

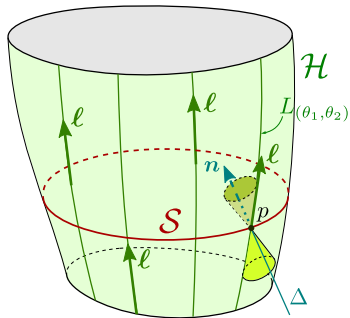


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Consequences

- Raychaudhuri equation implies **vanishing shear** σ_{ab} of ℓ
- Induced geometry on $\mathcal{H}$ is **invariant** along ℓ

Multipole moments of a NEH

Definition

The **shape** and **current** multipole moments I_{lm} and L_{lm} are defined by

$$I_{lm} + iL_{lm} \equiv - \oint_S \Psi_2 \dot{Y}_{lm} dS$$

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- The spherical harmonics $\dot{Y}_{lm}$ are defined **geometrically** from canonical polar coordinates (ϑ, φ)
- This definition is **independent** of the choice of $\mathcal{S}$

Horizon multipoles of a Kerr black hole

- Due to the **axisymmetry** of the Kerr metric, all non-zero multipoles have $m = 0$. The **reflection symmetry** accross the equatorial plane further implies

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- Moreover, in the regime of **small-spin** values $0 < \chi \ll 1$, the non-zero horizon multipoles behave as

$$I_{l0} + iL_{l0} \sim \alpha_l (i\chi)^l$$

$\hookrightarrow$ **same scaling** as normalized field multipoles $Z_l = (i\chi)^l$

Multipole moments of a *perturbed* NEH

- For a closed surface $\mathcal{S}$ embedded in spacetime, the invariant information is encoded into the **complex curvature**

$$\mathcal{K} \equiv -\frac{1}{4}\mathcal{R} + i\mathcal{X} \stackrel{\mathcal{S}}{=} \Psi_2 + \rho\rho' - \sigma\sigma'$$

$\uparrow \quad \uparrow$
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- Accordingly, the **shape** and **current** multipole moments I_{lm} and L_{lm} of $\mathcal{S}$ are defined by

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- The linear perturbations δI_{lm} and δL_{lm} are **gauge invariant**
 $\hookrightarrow$ geometrical definition of black hole shape Love numbers

Take-home messages

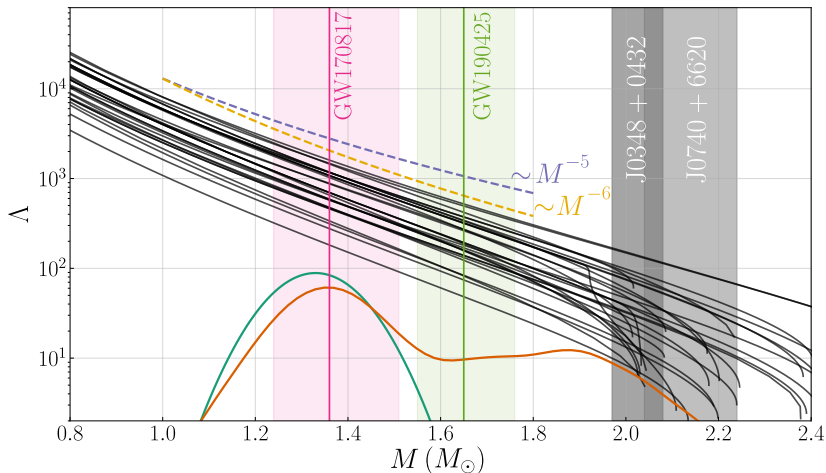
- 1 Black hole *field* Love numbers **do not vanish** in general
- 2 Kerr black holes have a **purely dissipative** (no conservative) tidal response
- 3 Horizon multipoles provide a **geometric characterization** of black hole *shape* Love numbers
- 4 Gravitational-wave observations enable **precision tests** of:
 - the no-hair theorem (stationary geometry)
 - tidal deformability (dynamical response)

From geometry to observable signatures

Additional Material

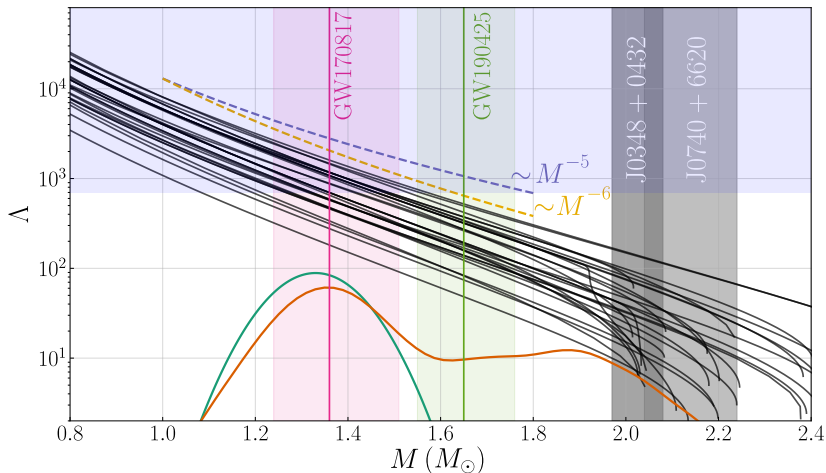
First constraints from GW observations

$$\Lambda \equiv \lambda_2/M^5 \propto k_2/C^5$$



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Geroch-Hansen multipole moments

- Consider a **stationary** spacetime with **timelike** Killing field ξ^a
- Define the norm λ and twist ω_a of ξ^a according to

$$\lambda \equiv -\xi^a \xi_a \quad \text{and} \quad \omega_a \equiv \varepsilon_{abcd} \xi^b \nabla^c \xi^d$$

- In **vacuum**, one can shown that

$$\nabla_{[a} \omega_{b]} = 0, \quad \text{implying} \quad \omega_a = \nabla_a \omega$$

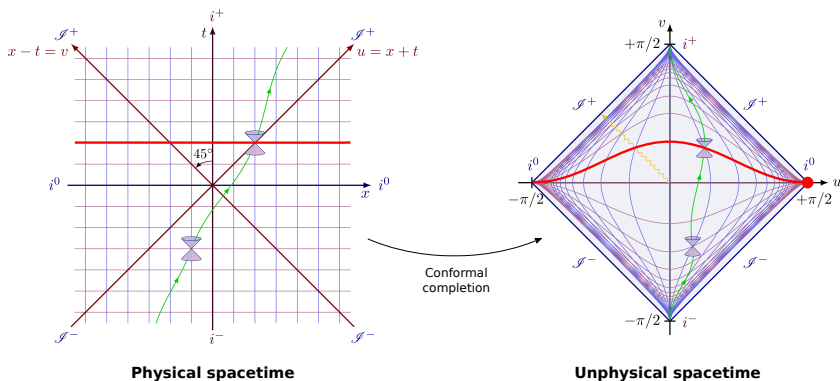
- Define a **complex potential** $\Phi = \Phi_M + i\Phi_S$, with

$$\Phi_M \equiv \frac{1 - \lambda^2 - \omega^2}{4\lambda} \quad \text{and} \quad \Phi_S \equiv \frac{\omega}{2\lambda}$$

$\hookrightarrow$ relativistic generalization of the Newtonian potential U

Geroch-Hansen multipole moments

Idea: define **mass-type** and **current-type** multipoles M_{lm} and S_{lm} from the potential Φ at **spatial infinity** i^0



Field multipoles of a Kerr black hole

- For a Kerr black hole of mass M and spin $S = Ma = M^2 \chi$, the norm and twist read (using BL coordinates)

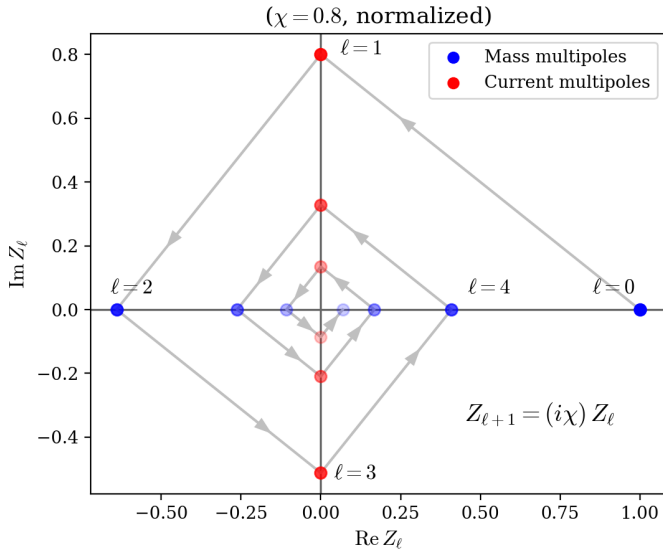
$$\lambda = \frac{r^2 - 2Mr + a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} \quad \text{and} \quad \omega = \frac{2Ma \cos \theta}{r^2 + a^2 \cos^2 \theta}$$

- Due to the **axisymmetry** of the Kerr metric, the non-zero multipoles all have $m = 0$. They read

$$\boxed{M_{l0} + iS_{l0} = M(\mathrm{i}a)^l} \quad \Longleftrightarrow \quad Z_l = (\mathrm{i}\chi)^l$$

Remark: the Kerr black hole is not merely **oblate**; it is also infinitely structured, yet completely **rigid**

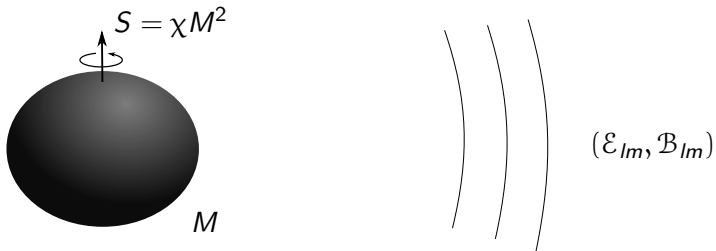
Field multipoles of a Kerr black hole



Do BHs have zero field Love numbers?

Reference	Background	Tidal field
[Binnington & Poisson 2009]	Schwarzschild	weak, generic l
[Damour & Nagar 2009]	Schwarzschild	weak, generic l
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $l = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(S^2)$	weak, $(l, m) = (2, 0)$
[Le Tiec & Casals 2021]	Exact Kerr	weak, generic l
[Chia 2021]	Exact Kerr	weak, generic l
[Goldberger et al. 2021]	Exact Kerr	weak, generic l
[Charalambous et al. 2021]	Exact Kerr	weak, generic l
[Hui 2021]	Schwar-(A)dS	weak, generic l

Investigating Kerr's Love



$$(\mathcal{E}_{lm}, \mathcal{B}_{lm}) \rightarrow \psi_0 \rightarrow \Psi \rightarrow h_{\alpha\beta} \rightarrow (M_{lm}, S_{lm}) \rightarrow \lambda_{lm}^{M/S, \mathcal{E}/\mathcal{B}}$$

Metric reconstruction through the Hertz potential Ψ

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c \rightarrow \infty} \psi_0^{lm} \propto \mathcal{E}_{lm} r^{l-2} [1 + 2k_{lm} (R/r)^{2l+1}] {}_2Y_{lm}(\theta, \phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{lm} \propto [\mathcal{E}_{lm} + \frac{3i}{l+1} \mathcal{B}_{lm}] R_{lm}(r) {}_2Y_{lm}(\theta, \phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{lm}(r) = \underbrace{r^{l-2} (1 + \dots)}_{\text{tidal field } R_{lm}^{\text{tidal}}} + k_{lm} \underbrace{r^{-l-3} (1 + \dots)}_{\text{linear response } R_{lm}^{\text{resp}}}$$

Why analytic continuation?

$$R_{lm}(r) = \underbrace{r^{l-2} (1 + \dots)}_{\text{tidal field } R_{lm}^{\text{tidal}}} + k_{lm} \underbrace{r^{-l-3} (1 + \dots)}_{\text{linear response } R_{lm}^{\text{resp}}}$$

Ambiguity in the linear response [Fang & Lovelace 2005; Gralla 2018]

The decaying solution R_{lm}^{resp} is affected by a radial coord. transfo.

Ambiguity in the tidal field [Pani, Gualtieri, Maselli & Ferrari 2015]

The growing solution $R_{lm}^{\text{tidal}} + \alpha R_{lm}^{\text{resp}}$ still qualifies as a tidal solution

Kerr black hole linear response

$$R_{lm}(r) = \underbrace{R_{lm}^{\text{tidal}}(r)}_{\sim r^{l-2}} + 2k_{lm} \underbrace{R_{lm}^{\text{resp}}(r)}_{\sim r^{-(l+3)}}$$

- The coefficients k_{lm} can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$$k_{lm} = -im\chi \frac{(l+2)!(l-2)!}{4(2l+1)!(2l)!} \prod_{n=1}^l [n^2(1-\chi^2) + m^2\chi^2]$$

- The linear response vanishes identically when:
 - the black hole spin vanishes ($\chi = 0$)
 - the tidal field is axisymmetric ($m = 0$)
- Reconstruct the Kerr black hole response $h_{\alpha\beta}^{\text{resp}}$ via Ψ^{resp}

Tidal Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\text{elec}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\text{mag}} \mathcal{B}_{ab}$$

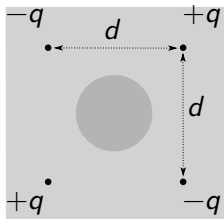
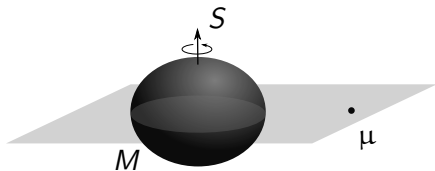
- For a **spinning black hole** we have the more general **tensorial** relations

$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}^{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}^{cd}$$

- To *linear order* in the black hole spin vector **S^a** we find

$$\begin{aligned} \delta M_{ab} &\doteq \frac{16}{45} M^3 \mathbf{S}^c \mathcal{E}^d{}_{\langle a} \mathcal{E}_{b \rangle cd} \\ \delta S_{ab} &\doteq \frac{16}{45} M^3 \mathbf{S}^c \mathcal{B}^d{}_{\langle a} \mathcal{E}_{b \rangle cd} \end{aligned}$$

Newtonian static quadrupolar tide



$$\mathcal{E}_{ab} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\delta M_{ab} \doteq 3Q \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↑

$$\frac{\chi}{180} (2M)^5 \frac{\mu}{r^3} = qd^2$$

Non-expanding horizons (NEHs)

Main properties

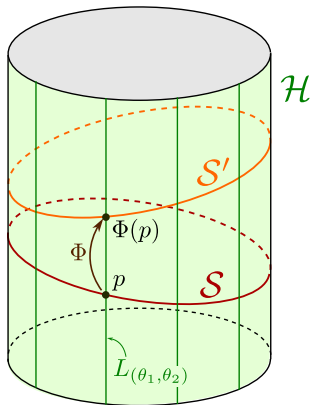
- All cross-sections are **isometric**
 $\hookrightarrow$ well-defined surface area

$$A \equiv \oint_S \varepsilon$$

- **Rotation 1-form** ω_a defined by

$$\mathcal{D}_a \ell^b \equiv \omega_a \ell^b,$$

with $\mathcal{D}_a$ the **intrinsic connection** induced on $\mathcal{H}$



Coulomb-type Weyl curvature scalar

- Given a complex null tetrad $(\ell, \mathbf{n}, \mathbf{m}, \mathbf{m}^*)$, the Coulomb-type curvature information is encoded into the **Weyl scalar**

$$\Psi_2 \equiv C_{abcd} \ell^a m^b m^{*c} n^d$$

- On a NEH $\mathcal{H}$, this curvature scalar is **frame-invariant**, with

$$\operatorname{Re} \Psi_2 \stackrel{\mathcal{H}}{=} -\frac{1}{4} \mathcal{R} \quad \longrightarrow \quad \textbf{shape} \text{ (intrinsic curvature)}$$

$$(\operatorname{Im} \Psi_2) \varepsilon_{ab} \stackrel{\mathcal{H}}{=} \mathcal{D}_{[a} \omega_{b]} \quad \longrightarrow \quad \textbf{rotational structure} \text{ (vorticity)}$$

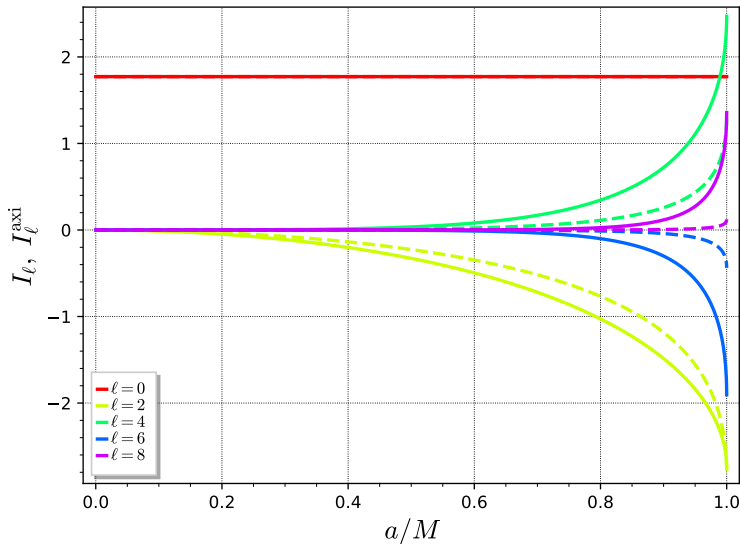
$\hookrightarrow$ natural candidate for horizon multipoles on $\mathcal{S}$

Effective sources for horizon multipoles

	Electrostatics	Horizon (shape)
thin shell	Volume $Q_{lm} = \int_{\mathbb{R}^3} r^l \rho Y_{lm} dV$	
	Surface $Q_{lm} \propto \oint_{\mathbb{S}^2} \tilde{\rho} Y_{lm} dS$	$I_{lm} \propto \oint_S \mathcal{R} Y_{lm} dS$
	Source Surface charge density $\tilde{\rho}$	Intrinsic curvature $\mathcal{R}$

	Magnetostatics	Horizon (current)
Volume	$\mu_{lm} = \int_{\mathbb{R}^3} r^l \varepsilon^{AB} j_B D_A Y_{lm} dV$	
Surface	$\mu_{lm} \propto \oint_{\mathbb{S}^2} \varepsilon^{AB} D_A \tilde{j}_B Y_{lm} dS$	$L_{lm} \propto \oint_S \varepsilon^{AB} D_A \omega_B Y_{lm} dS$
Source	Curl of surface current $\tilde{j}_A$	Curl of rotation 1-form ω_A

Shape multipoles of a Kerr black hole



Current multipoles of a Kerr black hole

