II. Post-Minkowskian theory

Expansion of the Einstein equation in powers of G, which measures the strength of the gravitational field.

Applies to weak fields: planets, main-sequence stars, white dwarfs

But also not-so-weak fields: black hole and neutron star binaries provided that the mutual gravitational attraction is weak enough.

Landau - Lifshitz formulation of GR

Instead of working with the metric $g_{\alpha\beta}$, we work with the gothic inverse ~~metric~~

$$\boxed{\mathfrak{g}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta}}$$ a tensor density

$$\det(\mathfrak{g}^{\alpha\beta}) = (\sqrt{-g})^4 \underbrace{\det(g^{\alpha\beta})}_{=\frac{1}{g}} = g$$

$$\downarrow$$

$$g^{\alpha\beta} = \frac{\mathfrak{g}^{\alpha\beta}}{\sqrt{-\det(\mathfrak{g}^{\alpha\beta})}} \longrightarrow g_{\alpha\beta}$$

The Einstein equation will be reformulated in terms of

$$\boxed{H^{\alpha\mu\beta\nu} \equiv \mathfrak{g}^{\alpha\beta}\,\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\,\mathfrak{g}^{\beta\mu}}\,,$$

which has the same symmetries as the Riemann tensor:

$$\begin{cases} H^{\mu\alpha\beta\nu} = -H^{\alpha\mu\beta\nu} \\ H^{\alpha\mu\nu\beta} = -H^{\alpha\mu\beta\nu} \\ H^{\beta\nu\alpha\mu} = H^{\alpha\mu\beta\nu} \end{cases}$$

irrespective of the Einstein equation

Remarkably, this tensor density satisfies the identity

$$\partial_\mu\partial_\nu H^{\alpha\mu\beta\nu} = 2(-g)\,\underset{\text{Einstein tensor}}{G^{\alpha\beta}} + \frac{16\pi G}{c^4}\underbrace{(-g)\,t^{\alpha\beta}_{LL}}_{\text{Landau-Lifshitz pseudotensor}}$$

where

$$(-g)\,t^{\alpha\beta}_{LL} \equiv \frac{c^4}{16\pi G}\Big\{ \partial_\lambda \mathfrak{g}^{\alpha\beta}\,\partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\,\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu}\,\partial_\rho \mathfrak{g}^{\lambda\nu}\,\partial_\nu \mathfrak{g}$$

$$- g^{\alpha\lambda} g_{\mu\nu}\,\partial_\rho \mathfrak{g}^{\beta\nu}\,\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda} g_{\mu\nu}\,\partial_\rho \mathfrak{g}^{\alpha\nu}\,\partial_\lambda \mathfrak{g}^{\mu\rho}$$

$$+ g_{\lambda\mu}\, g^{\nu\rho}\,\partial_\nu \mathfrak{g}^{\alpha\lambda}\,\partial_\rho \mathfrak{g}^{\beta\mu}$$

$$+ \frac{1}{8}\left(2g^{\alpha\lambda} g^{\beta\mu} - g^{\alpha\beta} g^{\lambda\mu}\right)\left(2 g_{\nu\rho} g_{\sigma\tau} - g_{\rho\sigma} g_{\nu\tau}\right)\partial_\lambda \mathfrak{g}^{\nu\tau}\,\partial_\mu \mathfrak{g}^{\rho\sigma}\Big\}$$

$$\sim \underline{\underline{\partial\mathfrak{g}\,\partial\mathfrak{g}}}$$

The Einstein equation, $G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$, can then be reformulated as

$$\boxed{\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right)}$$

<u>exact reformulation</u>

~ gravitational field energy-momentum (pseudo)tensor

From the antisymmetry in $\beta \leftrightarrow \nu$ we get

$$\partial_\beta \partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 0 \quad \Longrightarrow \quad \boxed{\partial_\beta\left[(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right)\right]}$$

ordinary conservation of the total energy-momentum pseudotensor

$$\Updownarrow \quad \text{(long calculation)}$$

$$\boxed{\nabla_\beta T^{\alpha\beta} = 0}$$

<u>Note</u>: any coordinate transformation that leaves g invariant is such that $\mathfrak{g}^{\alpha\beta}$ and $\partial_\lambda \mathfrak{g}^{\alpha\beta}$ transform as tensor fields

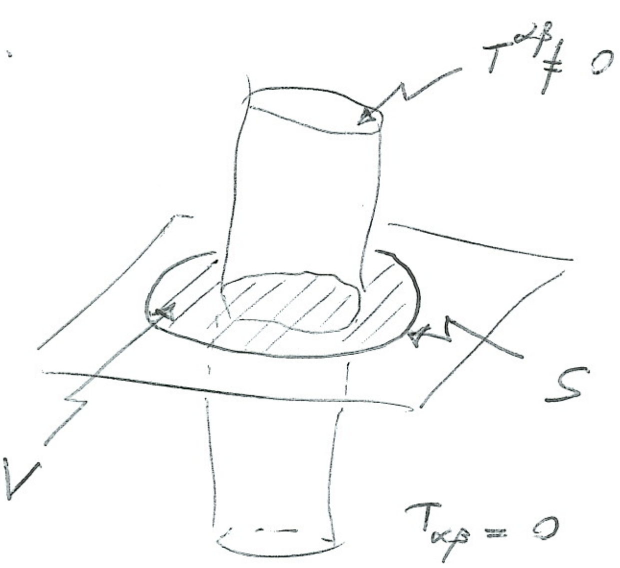
- Lorentz transformation $x^\alpha \longrightarrow \Lambda^\alpha{}_\beta\, x^\alpha$ with $\det \Lambda = 1$
- Uniform translations $x^\alpha \longrightarrow x^\alpha + c^\alpha$ with $c^\alpha = \text{cst}$

Integral conservation identities

$$\partial_\beta \left[(-g)\, T^{\alpha\beta} + t_{LL}^{\alpha\beta} \right] = 0 \quad \text{generalized the identity}$$

$\partial_\beta T^{\alpha\beta} = 0$ of linearized gravity.

Shocker!



Total momentum in V

$$\boxed{P^\alpha[V] \equiv \frac{1}{c} \int_V (-g)\left(T^{\alpha 0} + t_{LL}^{\alpha 0}\right) d^3x}$$

matter   grav. field

Total mass in V

$$\boxed{M[V] \equiv \frac{1}{c^2} \int_V (-g)\left(T^{00} + t_{LL}^{00}\right) d^3x}$$

Total three-momentum in V

$$\boxed{P^i[V] \equiv \frac{1}{c} \int_V (-g)\left(T^{i0} + t_{LL}^{i0}\right) d^3x}$$

$$P^\alpha = (Mc, p^i)$$

Thanks to the Einstein field equation, an alternative form in terms of surface integral can be given:

$$P^\alpha[V] = \frac{c^3}{16\pi G} \int_V \partial_\mu \partial_\nu H^{\alpha\mu 0\nu}\, d^3x = \frac{c^3}{16\pi G} \int_V \partial_k \left(\partial_\mu H^{\alpha\mu 0 k} \right) d^3x$$

because $H^{\alpha\mu 00} = 0$

The Gauss-Ostrogradsky theorem implies

$$\boxed{P^{\alpha}[V] = \frac{c^3}{16\pi G} \oint_S \partial_\mu H^{\alpha\mu 0 k}\, dS_k}$$

alternative definition for $P^{\alpha}[V]$

known given $g_{\alpha\beta}$

$$\boxed{M[V] = \frac{c^2}{16\pi G} \oint_S \partial_j H^{0j0k}\, dS_k}$$

because $H^{000k} = 0$

$$\boxed{P^i[V] = \frac{c^3}{16\pi G} \oint_S \partial_j H^{ij0k}\, dS_k - \frac{c^2}{16\pi G}\frac{d}{dt} \oint_S H^{0i0k}\, dS_k}$$

To obtain balance equations, we proceed as usual:

$$\frac{d}{dt} P^{\alpha}[V] = \int_V \partial_0 [(-g)(T^{\alpha 0} + t_{LL}^{\alpha 0})]\, d^3x$$

$$= -\int_V \partial_k [(-g)(T^{\alpha k} + t_{LL}^{\alpha k})]\, d^3x$$

$$= -\oint_S \cancel{(-g) T^{\alpha k}}\, dS_k - \oint_S (-g)\, t_{LL}^{\alpha k}\, dS_k$$

$$\boxed{\dot{P}^{\alpha}[V] = -\oint_S (-g)\, t_{LL}^{\alpha k}\, dS_k}$$

flux integral over t_{LL}

$$\boxed{\begin{aligned} \dot{M}[V] &= -\frac{1}{c}\oint_S (-g)\, t^{0k}_{LL}\, dS_k \\ \dot{P}^j[V] &= -\oint_S (-g)\, t^{jk}_{LL}\, dS_k \end{aligned}}$$

balance of mass-energy and three-momentum

In the limit where V is taken to include all of 3d space, $P^\alpha[V]$ is known to coincide with the ADM four-momentum P^α_{ADM} of an asymptotically flat spacetime, and its physical interpretation as a measure of total momentum is robust.

ADM mass

$$\boxed{\begin{aligned} M &\equiv \frac{1}{c^2}\int_{\text{all space}} (-g)(T^{00} + t^{00}_{LL})\, d^3x \\ &= \frac{c^2}{16\pi G}\oint_\infty \partial_j H^{0j0k}\, dS_k \end{aligned}}$$

ADM 3-momentum

$$\boxed{\begin{aligned} P^i &\equiv \frac{1}{c}\int_{\text{all space}} (-g)(T^{i0} + t^{i0}_{LL})\, d^3x \\ &= \frac{c^3}{16\pi G}\oint_\infty \partial_j H^{ij0k}\, dS_k - \frac{c^2}{16\pi G}\frac{d}{dt}\oint_\infty H^{0i0k}\, dS_k \end{aligned}}$$

Remarkably, the asymptotic behavior of the metric at large distances is enough to determine M and P^i. An intimate knowledge of the material source is not required.

Harmonic coordinates

Coordinate system (x^α). Each coordinate x^α can be viewed as a scalar field $X^{(\mu)}(x) = x^\mu$ on the manifold, such that $\partial_\alpha X^{(\mu)} = \delta^\mu_\alpha$.

The coordinates (x^α) are said to be harmonic iff $\boxed{\Box_g X^{(\mu)} = 0}$

generalization of Laplace's equation $\nabla^2 X = 0$

harmonic function

$g^{\alpha\beta} \nabla_\alpha \nabla_\beta$

$$\Box_g X^{(\mu)} = \nabla_\alpha X^{(\mu)\alpha} = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} X^{(\mu)}_\beta \right) = 0$$

where $X^{(\mu)}_\beta = \nabla_\beta X^{(\mu)} = \partial_\beta X^{(\mu)} = \delta^\mu_\beta$

$$\hookrightarrow \quad \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\mu} \right) = 0 \quad \longrightarrow \quad \boxed{\partial_\beta \mathfrak{g}^{\alpha\beta} = 0}$$

harmonic coordinate conditions

The harmonic gauge condition can <u>always</u> be imposed.

We seek $x'^{\mu} = f^{\mu}(x^{\alpha})$ such that

$$\partial_{\beta} \mathfrak{g}^{\alpha\beta} \neq 0 \quad \text{but} \quad \partial_{\nu'} \mathfrak{g}^{\mu'\nu'} = 0$$

$$\frac{1}{\sqrt{-g'}} \partial_{\nu'} \mathfrak{g}^{\mu'\nu'} = \Box_{g'} f^{\mu'} = \Box_g f^{\mu'}(x)$$

scalar operator

One can always find four functions $f^{\mu}(x)$ such that $\Box_g f^{\mu}(x) = 0$

Introduce the metric potentials $\boxed{h^{\alpha\beta} \equiv \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}}$

where $\eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$
in Lorentzian coordinates (x^{α})

deviation of gothic metric from flat spacetime

Then $\partial_{\beta} \mathfrak{g}^{\alpha\beta} = 0 \iff \boxed{\partial_{\beta} h^{\alpha\beta} = 0}$

harmonic gauge conditions

<u>Note</u>:
$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$$
$$g = \det(\eta^{\alpha\beta} - h^{\alpha\beta}) = -1 + h + \cdots \text{ when } h \equiv \eta_{\alpha\beta} h^{\alpha\beta} = \eta^{\alpha\beta} h_{\alpha\beta}$$
$$\sqrt{-g} = 1 - \frac{1}{2} h + \cdots$$
$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + \frac{1}{2}\eta^{\alpha\beta} h + \cdots = \eta^{\alpha\beta} - \bar{h}^{\alpha\beta} + \cdots$$
$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h + \cdots = \eta_{\alpha\beta} + \bar{h}^{\alpha\beta} + \cdots$$

Back to the balance equations

Energy balance :
$$\boxed{\frac{dE}{dt} = -\mathcal{P}}$$

where
$$\begin{cases} E = Mc^2 = \int (-g)(T^{00} + t_{LL}^{00})\, d^3x \\ \mathcal{P} = c \oint_\infty (-g)\, t_{LL}^{0k}\, dS_k \end{cases}$$

Momentum balance :
$$\boxed{\frac{dP^j}{dt} = -\mathcal{P}^j}$$

where
$$\begin{cases} P^j = \frac{1}{c} \int (-g)(T^{j0} + t_{LL}^{j0})\, d^3x \\ \mathcal{P}^j = \oint_\infty (-g)\, t_{LL}^{jk}\, dS_k \end{cases}$$

Good
- expressions for radiative fluxes $\mathcal{P}$, $\mathcal{P}^j$, ...
- expressions / definitions for E, P^j, ...
- balance equations describing how $(E, P^j, ...)$ change due to GW emission

Bad
- those definitions are not unique

How are those (P, P^j) related to those we introduced from the Isaacson effective stress-energy tensor ?

In the wave zone, we have $h^{\alpha\beta} = \frac{1}{r} f^{\alpha\beta}(\tau, \vec{n}) + O\left(\frac{1}{r^2}\right)$

$$\hookrightarrow \begin{cases} \partial_0 h^{\alpha\beta} = \frac{1}{c} \dot{h}^{\alpha\beta} + O\left(\frac{1}{r^2}\right) \\ \partial_j h^{\alpha\beta} = -\frac{n_j}{c} \dot{h}^{\alpha\beta} + O\left(\frac{1}{r^2}\right) \end{cases}$$

$$\hookrightarrow \boxed{\partial_\mu h^{\alpha\beta} = -\frac{1}{c} k_\mu \dot{h}^{\alpha\beta} + O\left(\frac{1}{r^2}\right)}$$

where $k_\mu \equiv (-1, \vec{n})$ is null : $\eta^{\alpha\beta} k_\alpha k_\beta = 0$

Inserting this expression into the LL pseudo-tensor $(-g) t_{LL}^{\alpha\beta} \sim \partial h \, \partial h$, and imposing the harmonic gauge conditions $\partial_\beta h^{\alpha\beta} = 0$, we find to leading order in $1/r$:

$$\boxed{(-g)\, t_{LL}^{\alpha\beta} = \frac{c^2}{32\pi G} \left(\dot{h}^{jk}_{TT} \, \dot{h}^{TT}_{jk} \right) k^\alpha k^\beta}$$

where $k^\alpha \equiv (1, \vec{n})$ and $h^{jk}_{TT} = (TT)^{jk}{}_{pq} \, h^{pq}$

Inserting this into the definitions of $\mathcal{P}$ and $\mathcal{P}^i$ we find

$$\begin{cases} \mathcal{P} = \lim\limits_{r\to\infty} \dfrac{c^3 r^2}{32\pi G} \oint d\Omega \left(\dot{h}_{ij}^{TT} \dot{h}^{ij}_{TT} \right) \\ \mathcal{P}^i = \lim\limits_{r\to\infty} \dfrac{c^2 r^2}{32\pi G} \oint d\Omega\, n^i \left(\dot{h}_{jk}^{TT} \dot{h}^{jk}_{TT} \right) \end{cases}$$

These expressions agree with those derived earlier from the Isaacson effective stress-energy tensor, in an average sense.

Relaxed Einstein equations

Thanks to the harmonic gauge condition $\partial_\beta h^{\alpha\beta} = 0$, the LHS of the Einstein equation becomes

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = -\underbrace{\Box}_{\eta^{\mu\nu}\partial_\mu\partial_\nu} h^{\alpha\beta} + h^{\mu\nu} \partial_\mu \partial_\nu h^{\alpha\beta} - \partial_\mu h^{\alpha\nu} \partial_\nu h^{\beta\mu}$$

$$\boxed{\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta}}$$

where $\tau^{\alpha\beta} = (-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta} + t_H^{\alpha\beta} \right)$

effective energy-momentum pseudotensor

$$(-g) t_H^{\alpha\beta} = \frac{c^4}{16\pi G} \left(\partial_\mu h^{\alpha\nu} \partial_\nu h^{\beta\mu} - h^{\mu\nu} \partial_{\mu\nu} h^{\alpha\beta} \right)$$

$$\tau^{\alpha\beta} = (-g)\left(T^{\alpha\beta}[\Psi, g] + t_{LL}^{\alpha\beta}[h] + t_{H}^{\alpha\beta}[h] \right)$$

↑ all the matter fields

ex : $T^{\alpha\beta} = (\varepsilon + p)\, u^\alpha u^\beta + p\, g^{\alpha\beta}$

$\Psi = \{ \varepsilon, p, u^\alpha \}$

$$\partial_\beta \Box h^{\alpha\beta} = \Box \partial_\beta h^{\alpha\beta} = -\frac{16\pi G}{c^4} \partial_\beta \tau^{\alpha\beta}$$

$$\hookrightarrow \boxed{\partial_\beta h^{\alpha\beta} = 0 \implies \partial_\beta \tau^{\alpha\beta} = 0}$$ (in fact $\Leftrightarrow$)

$$\Updownarrow$$

$$\nabla_\beta T^{\alpha\beta} = 0 \iff \partial_\beta \left[(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta} \right) \right] = 0$$

because $\partial_\beta \left[(-g)\, t_H^{\alpha\beta} \right] = 0$ identically

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$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \iff \begin{cases} \Box h^{\alpha\beta} = -\dfrac{16\pi G}{c^4} \tau^{\alpha\beta} \\ \partial_\beta h^{\alpha\beta} = 0 \iff \partial_\beta \tau^{\alpha\beta} = 0 \end{cases}$$

<u>exact reformulation</u>

- $\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta} \longrightarrow f^{\alpha\beta}[\Psi]$

  matter tells spacetime how to curve

- $\partial_\beta \tau^{\alpha\beta} = 0 \longrightarrow$ EOM for Ψ in $h^{\alpha\beta}$

  spacetime tells matter how to move

Solving for the field equation $\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta}$ <u>without</u> imposing the harmonic gauge conditions $\partial_\beta h^{\alpha\beta} = 0$ is solving the <u>relaxed Einstein field equation</u>.

Neglecting all terms $O(h^2)$ or higher, we have

$$\begin{cases} \Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} T^{\alpha\beta} \\ \partial_\beta h^{\alpha\beta} = 0 \iff \partial_\beta T^{\alpha\beta} = 0 \end{cases}$$

Since $h^{\alpha\beta} = \bar{h}^{\alpha\beta}_{\text{chap. I}}$, we recover the linearized theory

Iteration of the relaxed Einstein equation

For a given choice of matter variables, we solve the field equations (and the gauge condition) by successive approximations.

To construct the spacetime metric, we consider a formal expansion of the type

$$h^{\alpha\beta} = G k_1^{\alpha\beta} + G^2 k_2^{\alpha\beta} + G^3 k_3^{\alpha\beta} + \cdots$$

post-Minkowskian expansion

$$h_0^{\alpha\beta} = 0$$

$$h_1^{\alpha\beta} = G k_1^{\alpha\beta}$$

$$h_2^{\alpha\beta} = G k_1^{\alpha\beta} + G^2 k_2^{\alpha\beta}$$

$$\vdots$$

The metric can be reconstructed as

$$\begin{cases} g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta} + h_{\alpha\gamma} h^{\gamma}{}_{\beta} - \frac{1}{2} h h_{\alpha\beta} \\ \qquad + \left(\frac{1}{8} h^2 - \frac{1}{4} h^{\mu\nu} h_{\mu\nu}\right) \eta_{\alpha\beta} + O(G^3) \\ g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + \frac{1}{2} h \eta^{\alpha\beta} - \frac{1}{2} h h^{\alpha\beta} + \left(\frac{1}{8} h^2 + \frac{1}{4} h^{\mu\nu} h_{\mu\nu}\right) \eta^{\alpha\beta} + O(G^3) \\ \sqrt{-g} = 1 - \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} + O(G^3) \end{cases}$$

$$h_0^{\alpha\beta} = 0 \longrightarrow g^0_{\alpha\beta} = \eta_{\alpha\beta} \longrightarrow \begin{cases} T^{\alpha\beta}[\Psi, g] = T^{\alpha\beta}[\Psi, \eta] \\ t_{LL}^{\alpha\beta}[h] = t_{LL}^{\alpha\beta}[h_0] = 0 \\ t_H^{\alpha\beta}[h] = t_H^{\alpha\beta}[h_0] = 0 \end{cases}$$

$$\downarrow$$

$\boxed{1^{st} \text{ iteration}}$

solve $\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau_0^{\alpha\beta} \longleftarrow \tau_0^{\alpha\beta} = T^{\alpha\beta}[\Psi, \eta]$

for $h_1^{\alpha\beta}[\Psi] = G k_1^{\alpha\beta}$

$$\downarrow$$

$$g^1_{\alpha\beta} = \eta_{\alpha\beta} + h^1_{\alpha\beta} - \frac{1}{2} h^1 \eta_{\alpha\beta}$$

$$\downarrow$$

$$\tau_1^{\alpha\beta} \sim T^{\alpha\beta}[\Psi, g_1] + t_{LL}^{\alpha\beta}[h_1] + t_H^{\alpha\beta}[h_1] \sim \partial h_1 \partial h_1$$

$$\downarrow$$

solve $\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau_1^{\alpha\beta}$

for $h_2^{\alpha\beta}[\Psi] = G k_1^{\alpha\beta} + G^2 k_2^{\alpha\beta}$ $\boxed{2^{nd} \text{ iteration}}$

$$\downarrow$$

$$g^2_{\alpha\beta} = \eta_\alpha + h^2_{\alpha\beta} - \frac{1}{2} h^2 \eta_{\alpha\beta} + h_{\alpha\gamma} h^\gamma_{2\beta} - \frac{1}{2} h^2 h^2_{\alpha\beta} + \left(\frac{1}{8} h_2^2 - \frac{1}{4} h_2^{\mu\nu} h^2_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3)$$

$$\downarrow$$

$$\tau_2^{\alpha\beta} \sim T^{\alpha\beta}[\Psi, g_2] + t_{LL}^{\alpha\beta}[h_2] + t_H^{\alpha\beta}[h_2] \sim \partial h_2 \partial h_2$$

$$\downarrow$$

...

After n iterations we obtain

$$h_n^{\alpha\beta}[\Psi] = G k_1^{\alpha\beta} + G^2 k_2^{\alpha\beta} + \dots + G^n k_n^{\alpha\beta}$$

To determine Ψ we implement the gauge condition / conservation statement

$$\partial_\beta h_n^{\alpha\beta} = 0 \iff \partial_\beta \tau_{n-1}^{\alpha\beta} = 0$$

This determines Ψ and returns $g_{\alpha\beta}^n(x)$ as a proper tensor field in spacetime.

Example : $\Psi = \{ \vec{r}_1(t), \dots, \vec{r}_N(t) \}$ for N point masses

(1) n iterations of the relaxed Einstein equation $\rightarrow$ $h_n^{\alpha\beta}(x; \vec{r}_A(t))$

(2) conservation equation $\partial_\beta \tau_{n-1}^{\alpha\beta} = 0$

$$\hookrightarrow \quad \frac{d^2\vec{r}_A}{dt^2} = O(G) + O(G^2) + \dots + O(G^{n-1})$$

$$\downarrow$$

$$\vec{r}_A(t) \text{ known}$$

(!) To obtain the Newtonian equations of motion, two iterations of the relaxed field equations are required.

Integration of the wave equation

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta}[\Psi, g] + t_{LL}^{\alpha\beta}[h] + t_{H}^{\alpha\beta}[h] \right)$$

is highly non-linear !

However $\Box h_n^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau_{n-1}^{\alpha\beta}$ is linear in $h_n^{\alpha\beta}$.

We wish to solve for the wave equation

$$\boxed{\Box \psi = -4\pi \mu}$$

$\psi \sim h^{\alpha\beta}$ $\qquad$ $\mu \sim \frac{4G}{c^4}\tau^{\alpha\beta}$

⚠ μ does not have compact support

From the knowledge of the retarded Green's function in flat spacetime, the solution is known as

$$\boxed{\psi(t,\vec{x}) = \int_{\mathcal{C}} \frac{\mu\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|} d^3x'}$$

Integration over the past lightcone $\mathcal{C}(x)$ of the field point $x^\alpha = (t, \vec{x})$

Near zone and wave zone: an example

Oscillating dipole $\mu(t, \vec{x}) = -\vec{p} \cdot \vec{\nabla} \delta(\vec{x}) \cos\omega t$

constant direction

oscillation at frequency $f = \frac{\omega}{2\pi}$

$\downarrow$

Scalar radiation with potential $\psi(t, \vec{x}) = (\vec{p} \cdot \vec{n}) \left[\frac{\cos\omega(t - r/c)}{r^2} - \frac{\omega}{c} \frac{\sin\omega(t - r/c)}{r} \right]$

and wavelength $\lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$

- Near zone: $r \ll \lambda = \frac{2\pi c}{\omega}$

$$\frac{r}{\lambda\!\!\!/} = \frac{\omega r}{c} \ll 1 \rightarrow \begin{cases} \cos\omega\left(t - \frac{r}{c}\right) = \cos\omega t + \frac{\omega r}{c} \sin\omega t + O\left(\frac{r^2}{\lambda^2}\right) \\ \sin\omega\left(t - \frac{r}{c}\right) = \sin\omega t - \frac{\omega r}{c} \cos\omega t + O\left(\frac{r^2}{\lambda^2}\right) \end{cases}$$

$\downarrow$

$$\psi(t, \vec{x}) = (\vec{p} \cdot \vec{n}) \frac{\cos\omega t}{r^2} \left[1 + O\left(\frac{r^2}{\lambda^2}\right) \right]$$

$$\frac{\partial_t \psi}{c |\vec{\nabla}\psi|} = O\left(\frac{r}{\lambda}\right) \ll 1$$

$\hookrightarrow$ retardations are small

- <u>Wave zone</u> : $r \gg \lambda = 2\pi c/\omega$

$t - r/c$

$$\frac{\lambda}{r} = \frac{c}{\omega r} \ll 1 \longrightarrow \psi(t,\vec{x}) = -(\vec{p}\cdot\vec{n})\,\frac{\omega}{c}\,\frac{\sin\omega\tau}{r}\left[1 + O\left(\frac{\lambda}{r}\right)\right]$$

$$\frac{\partial_t \psi}{c|\vec{\nabla}\psi|} = O(1)$$

↳ <u>retardation cannot be neglected</u>

Near zone / wave zone and the slow-motion condition

- $t_c \equiv$ characteristic timescale of the source $\quad \boxed{\partial_t \mu \sim \frac{\mu}{t_c}}$
- $\omega_c \equiv \frac{2\pi}{t_c} =$ characteristic frequency "
- $\lambda_c \equiv \frac{2\pi c}{\omega_c} = c t_c =$ characteristic wavelength of the radiation

$$\boxed{\begin{array}{ll} \underline{\text{Near zone}}: & r \ll \lambda_c = \frac{2\pi c}{\omega_c} = c t_c \\ \underline{\text{Wave zone}}: & r \gg \lambda_c = \frac{2\pi c}{\omega_c} = c t_c \end{array}}$$

In the near zone, $\frac{r}{c}\frac{\partial \mu}{\partial t} \approx \frac{r}{\lambda_c}\mu \ll \mu$

↳ source retardations are unimportant

Moreover, when the source μ has a compact-supported piece μ_c (for example $T^{\alpha\beta}$) then we can introduce

- $r_c \equiv$ characteristic size of μ_c
- $v_c \equiv \frac{r_c}{t_c} =$ characteristic velocity within μ_c

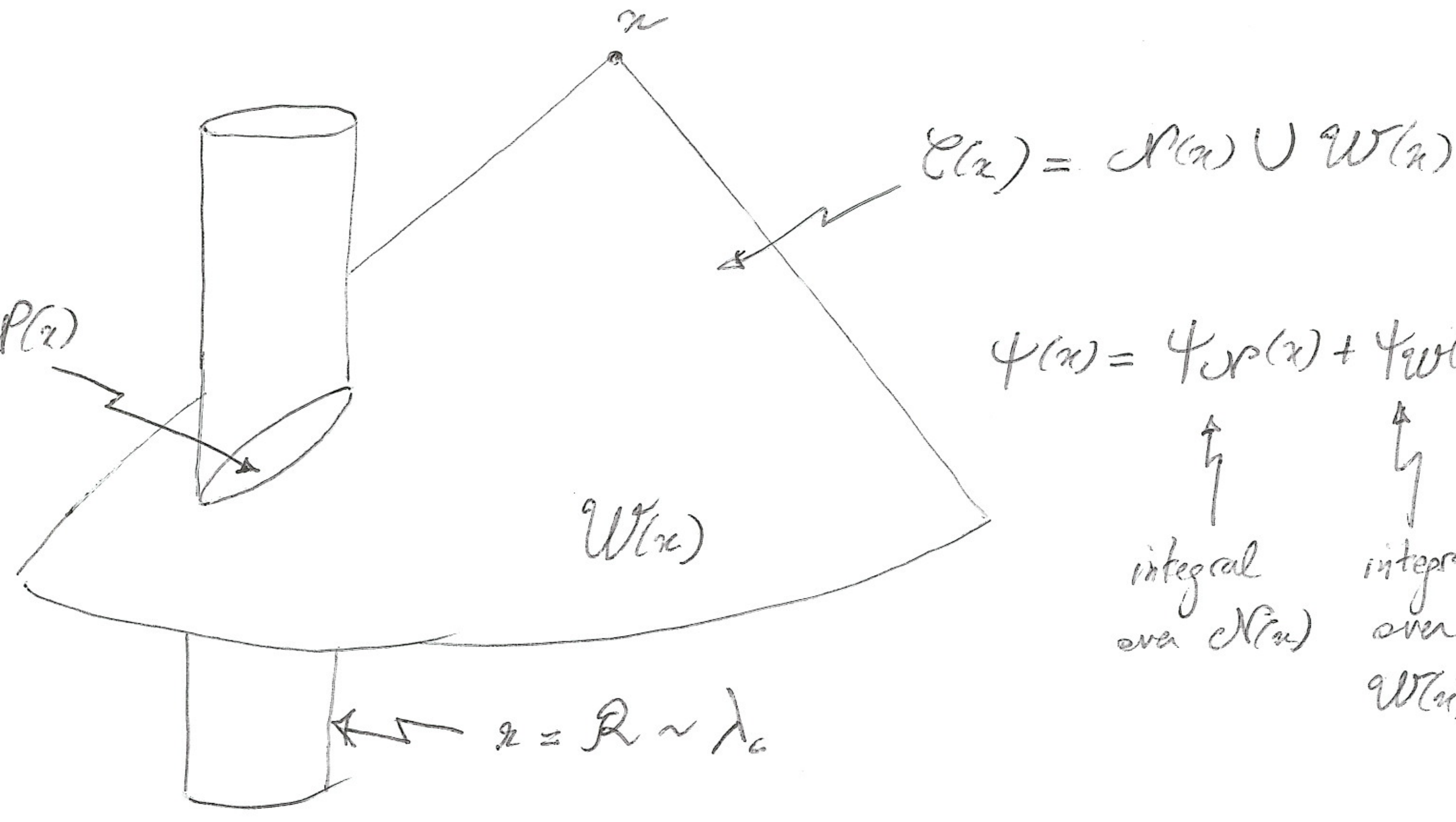
<u>slow-motion</u> <u>condition</u>	$v_c \ll c \iff r_c \ll \lambda_c$

$\Downarrow$

Post-Newtonian source

μ_c is deep in the near zone

<u>Integration domains</u>



$\psi(x) = \psi_{\mathcal{N}}(x) + \psi_{\mathcal{W}}(x)$

integral over $\mathcal{N}(x)$ — integral over $\mathcal{W}(x)$

Integration over the near zone

$$\psi_{\mathcal{N}}(x) = \int_{\mathcal{N}} \frac{\mu(t-|\vec{x}-\vec{x}'|/c\,,\,\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

Two cases: (1) $r = |\vec{x}| < R \Rightarrow x$ in the near zone

(2) $r = |\vec{x}| > R \Rightarrow x$ in the wave zone

$r \to \infty$

(1) Near-zone field point

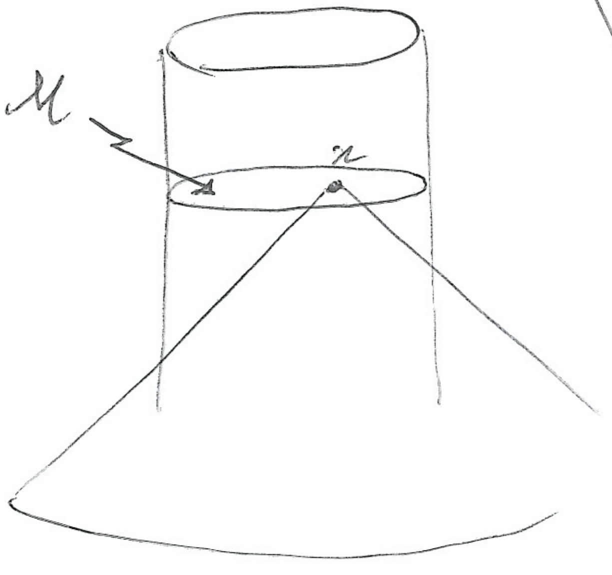
$\vec{x}$ and $\vec{x}'$ in the near zone, so $|\vec{x}-\vec{x}'|$ is "small"

$$\mu\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right) = \mu(t) - \frac{1}{c}\frac{\partial \mu}{\partial t}|\vec{x}-\vec{x}'| + \frac{1}{2c^2}\frac{\partial^2 \mu}{\partial t^2}|\vec{x}-\vec{x}'|^2 + \dots$$

$$\boxed{\psi_{\mathcal{N}}(t,\vec{x}) = \sum_{l=0}^{\infty} \frac{(-)^l}{l!\,c^l}\left(\frac{\partial}{\partial t}\right)^l \int_{\mathcal{M}} \mu(t,\vec{x}')\,|\vec{x}-\vec{x}'|^{l-1}\, d^3x'}$$

instantaneous potentials

$t = \text{cst}$ surface bounded by $r' = R$



PN expansion in powers of $\frac{1}{c}$

(2) Wave-zone field point

$$\frac{\mu\left(t-\frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|} = \int \underbrace{\frac{\mu\left(t-\frac{|\vec{x}-\vec{x}'|}{c}, \vec{y}\right)}{|\vec{x}-\vec{x}'|}}_{\equiv\, g(\vec{x},\vec{x}',\vec{y})} \delta(\vec{y}-\vec{x}')\, d^3y$$

Taylor-expand around $\vec{x}'=\vec{0}$:

$$g(\vec{x},\vec{x}',\vec{y}) = g(\vec{x},\vec{0},\vec{y}) + \frac{\partial g}{\partial x'^i} x'^i + \frac{1}{2}\frac{\partial^2 g}{\partial x'^i \partial x'^j} x'^i x'^j + \cdots$$

$$= g(\vec{x},\vec{0},\vec{y}) - \frac{\partial g}{\partial x^i} x'^i + \frac{1}{2}\frac{\partial^2 g}{\partial x^i \partial x^j} x'^i x'^j + \cdots$$

$$= g(r,\vec{0},\vec{y}) - \frac{\partial g(r,0,\vec{y})}{\partial x^i} x'^i + \frac{1}{2}\frac{\partial^2 g(r,\vec{0},\vec{y})}{\partial x^i \partial x^j} x'^i x'^j + \cdots$$

$$= \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} x'^L \partial_L g(r,0,\vec{y})$$

$\downarrow$

where $L \equiv j_1 j_2 \cdots j_\ell$ is a multi-index

$$\frac{\mu\left(t-\frac{|\vec{x}-\vec{x}'|}{c}, \vec{y}\right)}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} x'^L \partial_L\left(\frac{\mu(t-r/c, \vec{y})}{r}\right)$$

$\downarrow$

$$\boxed{\psi_{\mathcal{N}}(t,\vec{x}) = \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L\left[\frac{1}{r}\int_{\mathcal{M}} \mu(\overset{t-r/c}{\tau}, \vec{x}')\, x'^L d^3x'\right]}$$

$t=\tau$ surface bounded by $r=R$

This is valid everywhere in the wave zone.

In the far-away wave zone, as $r \to +\infty$, it reduces to

$$\Psi_{GW}(t,\vec{x}) = \frac{1}{r}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\int_{\mathcal{M}} \partial_L \mu(\tau,\vec{x}')\, x'^L d^3x' + O\left(\frac{1}{r^2}\right)$$

$$\partial_i \mu = -\frac{1}{c}\mu^{(1)}\partial_i r = -\frac{n_i}{c}\mu^{(1)}$$

$$\partial_i\partial_j \mu = \frac{n_i n_j}{c^2}\mu^{(2)} + O\left(\frac{1}{r}\right)$$

$\vdots$

$$\partial_L \mu = (-1)^l \frac{n_L}{c^l}\mu^{(l)} + O\left(\frac{1}{r}\right)$$

$\downarrow$

$$\boxed{\Psi_{GW}(t,\vec{x}) = \frac{1}{r}\sum_{l=0}^{\infty}\frac{n_L}{l!\,c^l}\mu_L^{(l)}(\tau)} + O\left(\frac{1}{r^2}\right)$$

where $\boxed{\mu_L(\tau) = \int_{\mathcal{M}} \mu(\tau,\vec{x}')\, x'^L d^3x'}$

<u>multiple expansion</u> of the potential Ψ_{GW}

<u>Note</u>: higher l-pole moments come with a higher power of $1/c$

Matter source : point particles

Point particle of mass m, following a worldline $x^\alpha = z^\alpha(\tau)$

$$\boxed{T^{\alpha\beta}(x) = mc \int u^\alpha u^\beta \, \delta_4(x,z)\, d\tau}$$

where $u^\alpha = \dfrac{dz^\alpha}{d\tau}$ is the particle's four-velocity, such that $u^\alpha u_\alpha = -c^2$,

and $\delta_4(x,z)$ is the four-dimensional, invariant Dirac distribution,

such that

$$\delta_4(x,z) = \frac{\delta(x^0 - z^0)\overbrace{\delta(x^1-z^1)\,\delta(x^2-z^2)\,\delta(x^3-z^3)}^{\delta(\vec{x}-\vec{z})}}{\sqrt{-g}}$$

$$\downarrow$$

fns of τ

$$T^{\alpha\beta}(x) = mc \int \frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}\, \frac{\delta(x^0 - z^0)\,\delta(\vec{x}-\vec{z})}{\sqrt{-g}}\, d\tau$$

fn of z^0

$$= mc \int \frac{dz^\alpha}{dz^0}\frac{dz^\beta}{dz^0}\frac{dz^0}{d\tau}\, \frac{\delta(x^0 - z^0)\,\delta(\vec{x}-\vec{z})}{\sqrt{-g}}\, dz^0$$

$$= mc\, \frac{dz^\alpha}{dz^0}\frac{dz^\beta}{dz^0}\frac{dz^0}{d\tau}\, \frac{\delta(\vec{x}-\vec{z})}{\sqrt{-g}} \qquad \text{where } z^0 = x^0 = ct$$

$$= \frac{m}{c}\, v^\alpha v^\beta \frac{dz^0}{d\tau}\, \frac{\delta(\vec{x}-\vec{z}(t))}{\sqrt{-g}} \qquad \text{where } v^\alpha = \frac{dz^\alpha}{dt}$$

Then $c^2 d\tau^2 = -g_{\alpha\beta}\, dz^\alpha dz^\beta$ implies $\frac{d\tau}{dz^0} = \frac{1}{c}\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}$

$$\boxed{T^{\alpha\beta}(t, \vec{x}) = \frac{m\, v^\alpha v^\beta}{\sqrt{-g}\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}}\, \delta(\vec{x} - \vec{z}(t))}$$

where $v^\alpha = \frac{dz^\alpha}{dt} = (c, \vec{v})$

of course $g_{\alpha\beta}$ and g are to be evaluated at the position $\vec{x} = \vec{z}(t)$ of the particle

$\longrightarrow$ <u>regularization</u> of the divergent self-fields

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The slow-motion condition $v_c \ll v$ implies a hierarchy between the components of $T^{\alpha\beta}$:

$$\frac{T^{0i}}{T^{00}} \sim \frac{v_c}{c} \quad , \quad \frac{T^{ij}}{T^{00}} \sim \left(\frac{v_c}{c}\right)^2$$

What determines the motion of each particle ?

$$\nabla_\beta T^{\alpha\beta} = 0 \quad \Longrightarrow \quad \frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma = 0$$

<u>geodesic</u> motion in a <u>regularized</u> metric

$$\Longrightarrow \quad \frac{du_\alpha}{d\tau} = - g_{\alpha\lambda} \Gamma^\lambda{}_{\beta\gamma} u^\beta u^\gamma$$

$$= \frac{1}{2} (\partial_\alpha g_{\beta\gamma}) u^\beta u^\gamma \quad \text{(Hamiltonian form)}$$

$$\begin{cases} u^\alpha = \dfrac{dt}{d\tau}\dfrac{dz^\alpha}{dt} = \dfrac{c\, v^\alpha}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}} \\[2ex] u_\alpha = g_{\alpha\lambda} u^\lambda = \dfrac{c\, g_{\alpha\lambda} v^\lambda}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}} \end{cases}$$

$$\cancel{\frac{c}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}}} \frac{d}{dt}\left(\frac{\cancel{c}\, g_{\alpha\lambda} v^\lambda}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}} \right) = \frac{1}{2} (\partial_\alpha g_{\beta\gamma}) \frac{\cancel{c^2}\, v^\beta v^\gamma}{\left(\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}\right)^{\cancel{2}}}$$

$$\boxed{\frac{dP^i}{dt} = F^i \quad \text{where} \quad \begin{cases} P^i = \dfrac{g_{i\alpha} v^\alpha}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}} \\[2ex] F^i = \dfrac{1}{2} \dfrac{(\partial_i g_{\alpha\beta}) v^\alpha v^\beta}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}} \end{cases}}$$

$$a^i \equiv \frac{dv^i}{dt} = F^i - \frac{d}{dt}\left(p^i - v^i\right)$$

computed from $g_{\alpha\beta}$ and $\partial_i g_{\alpha\beta}$ at the particle's location

<u>Example</u> : At Newtonian order,

$$\begin{cases} g_{00} = -1 + \dfrac{2U}{c^2} + O\left(\dfrac{1}{c^4}\right) \\ g_{0i} = O\left(\dfrac{1}{c^3}\right) \\ g_{ij} = \delta_{ij} + O\left(\dfrac{1}{c^2}\right) \end{cases}$$

$$\begin{cases} \sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2} = 1 + O\left(\dfrac{1}{c^2}\right) \\ g_{i\alpha} v^\alpha = g_{i0} v^0 + g_{ij} v^j = v^i + O\left(\dfrac{1}{c^2}\right) \\ (\partial_i g_{\alpha\beta}) v^\alpha v^\beta = (\partial_i g_{00}) c^2 + O\left(\dfrac{1}{c^2}\right) = 2\,\partial_i U + O\left(\dfrac{1}{c^2}\right) \end{cases}$$

$$\begin{cases} p^i = v^i + O\left(\dfrac{1}{c^2}\right) \\ F^i = \partial_i U + O\left(\dfrac{1}{c^2}\right) \end{cases}$$

$$a^i = \partial_i U + O\left(\frac{1}{c^2}\right)$$

Matter source = perfect fluid

$$T^{\alpha\beta} = \underbrace{(\varepsilon + p)}_{\text{proper density of total energy}} \frac{u^\alpha u^\beta}{c^2} + \underbrace{p}_{\text{pressure}}\, g^{\alpha\beta} \qquad \text{where} \qquad \varepsilon = \underbrace{\rho}_{\text{proper mass density}} c^2 + \underbrace{e_{int}}_{\text{density of internal (thermodynamic) energy}}$$

- Mass conservation : $\nabla_\alpha j^\alpha = 0$ where $j^\alpha = \rho u^\alpha$ is the mass current
- Energy-momentum conservation : $\nabla_\alpha T^{\alpha\beta} = 0$

$$\left\{ \begin{array}{l} \left. \begin{array}{l} \dfrac{d\rho}{d\tau} + \rho \nabla_\alpha u^\alpha = 0 \\[2ex] \dfrac{d\varepsilon}{d\tau} + (\varepsilon + p) \nabla_\alpha u^\alpha = 0 \end{array} \right\} \longrightarrow \rho \dfrac{de_{int}}{d\tau} = (e_{int} + p) \dfrac{d\rho}{d\tau} \quad \underline{\text{1st law of thermo}} \\[4ex] (\varepsilon + p) \dfrac{Du^\alpha}{d\tau} + \overbrace{c^2 \underbrace{\left(g^{\alpha\beta} + \dfrac{u^\alpha u^\beta}{c^2} \right)}_{\text{projector } \perp \text{ to } u} \nabla_\beta p}^{\text{force}} = 0 \quad \underline{\text{Euler eq.}} \end{array} \right.$$

Newtonian form:

$$u^\alpha = \gamma \underbrace{(c, \vec{v})}_{v^\alpha} \quad \text{where} \quad \gamma = \frac{u^0}{c} = \frac{1}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta / c^2}}$$

$$\nabla_\alpha j^\alpha = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g}\, j^\alpha) = 0 \quad \Rightarrow \quad \partial_\alpha (\sqrt{-g}\, \rho \gamma v^\alpha) = 0$$

$$\Downarrow$$

$$\boxed{\partial_t \rho_* + \partial_i (\rho_* v^i) = 0}$$

where $\rho_* \equiv \sqrt{-g}\, \gamma \rho = \sqrt{-g}\, \rho u^0 / c$