I. The linearized theory (weak fields)

Covariant components of the metric with respect to "almost Lorentzian" coord.:

$$\underbrace{g_{\alpha\beta} = \eta_{\alpha\beta}}_{\text{Minkowski metric}} + \underbrace{h_{\alpha\beta}}_{\text{metric perturbation}} \quad \text{where} \quad \begin{cases} \eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1) \\ |h_{\alpha\beta}| \ll 1 \end{cases}$$

always possible except

- near compact objects $h \sim \frac{GM}{c^2R} \lesssim 1$
- cosmological context: FLRW background

Linearized Einstein equation:

- Neglect all terms $O(h^2)$ or higher
- Lower and raise indices using $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\beta\partial_\gamma h_{\alpha\delta} - \partial_\beta\partial_\delta h_{\alpha\gamma} - \partial_\alpha\partial_\gamma h_{\beta\delta} + \partial_\alpha\partial_\delta h_{\beta\gamma}\right)$$

$$G_{\alpha\beta} = -\frac{1}{2}\left(\Box \bar{h}_{\alpha\beta} - 2\,\partial^\mu\partial_{(\alpha}\bar{h}_{\beta)\mu} + \eta_{\alpha\beta}\,\partial^\mu\partial^\nu \bar{h}_{\mu\nu}\right)$$

$\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ flat spacetime wave operator

where $\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$

with $h \equiv \eta^{\mu\nu}h_{\mu\nu} \implies \bar{h} = -h \Rightarrow h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h}$

One can easily check that $G^{\alpha\beta}$ obeys $\underline{\partial_\beta G^{\alpha\beta} = 0}$.

$\boxed{\text{EXO}}$

linearized Bianchi identity

Linearized Einstein equation :

$$\boxed{\Box \bar{h}_{\alpha\beta} - 2\, \partial^\mu \partial_{(\alpha} \bar{h}_{\beta)\mu} + \eta_{\alpha\beta}\, \partial^\mu \partial^\nu \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\alpha\beta}}$$

Consequently, we also have $\underline{\partial_\beta T^{\alpha\beta} = 0}$.

linearized energy-momentum conservation

$\Updownarrow$

conservation of energy-momentum with respect to the flat background $\eta_{\alpha\beta}$

Matter fields can exchange energy and momentum between each other, but they <u>do not exchange</u> energy and momentum with the <u>gravitational field</u>.
The linearized theory <u>cannot be applied</u> to systems (like stars) that are <u>bound</u> by gravitational forces.

Gauge transformation :

The decomposition $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$ is preserved by

$$h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} - 2\, \partial_{(\alpha} \xi_{\beta)}$$

generator such that $x'^\alpha = x^\alpha + \xi^\alpha(x)$

Lorenz gauge condition:

By analogy with the Lorenz gauge of EM, we wish to impose the four conditions

$$\boxed{\partial^\beta \bar{h}_{\alpha\beta} = 0}$$

Always possible! If $\partial^\beta \bar{h}_{\alpha\beta} \neq 0$ then by a gauge transfo we get

$$\partial^\beta \bar{h}'_{\alpha\beta} = \partial^\beta \bar{h}_{\alpha\beta} - \Box \xi_\alpha = 0 \iff \Box \xi_\alpha = \partial^\beta \bar{h}_{\alpha\beta}$$

$\hookrightarrow \xi_\alpha$

But not unique! Two Lorenz-gauge perturbation can be related by a generator ξ_α obeying $\Box \xi_\alpha = 0$

In the Lorenz gauge, the linearized Einstein equation becomes

$$\boxed{\Box \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}}$$

- Out of the ten original field $h_{\alpha\beta}$, only six are independent.
- It appears that all of those are radiative.

As we shall see this is not true.

Only two radiative dof.

This is a coordinate artefact

~~Lorenz gauge:~~

$$\partial^\beta \bar{h}_{\alpha\beta} = 0 \quad \Longrightarrow \quad \Box \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}$$

~~Always possible by a suitable gauge transformation~~

~~↳ Only six independent degrees of freedom out of ten.~~

~~⚠ Not all degrees of freedom are radiative → only two~~

Decomposition of the metric into irreducible pieces:

Flanagan & Hughes, New J. Phys 7 (2005) 204, gr-qc/0501041
Poisson & Will, Gravity, Chap. 5.5

Vector: $A^j = \partial^j A + A_T^j$ where $\partial_j A_T^j = 0$

longitudinal (1 dof) — transverse (2 dof)

$$\tilde{A}^j = i\tilde{A}k^j + \tilde{A}_T^j \quad \text{where} \quad ik_j \tilde{A}_T^j = 0$$

$$A^j(\vec{x}) = \int \tilde{A}^j(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d^3k$$

Symmetric Tensor:

$$B^{jk} = \underset{\substack{\uparrow \\ \text{trace} \\ (1\text{ dof})}}{\frac{1}{3}\delta^{jk} B} + \underset{\substack{\uparrow \\ \text{longitudinal - tracefree} \\ (1\text{ dof})}}{\left(\partial^j\partial^k - \frac{1}{3}\delta^{jk}\Delta\right) C}$$

$$+ \underset{\substack{\uparrow \\ \text{longitudinal - transverse} \\ (2\text{ dof})}}{2\,\partial^{(j} C_T^{k)}} + \underset{\substack{\uparrow \\ \text{transverse - tracefree} \\ (2\text{ dof})}}{C_{TT}^{jk}}$$

where
$$\begin{cases} \partial_j C_T^j = 0 \\ \partial_k C_{TT}^{jk} = 0 \\ \delta_{jk} C_{TT}^{jk} = 0 \end{cases}$$

$$\tilde{B}^{jk} = \frac{1}{3}\delta^{jk}\tilde{B} + \left(-k^j k^k + \frac{1}{3}\delta^{jk} k^2\right)\tilde{C}$$

$$+ 2ik^{(j}\tilde{C}_T^{k)} + \tilde{C}_{TT}^{jk}$$

where
$$\begin{cases} ik_j \tilde{C}_T^{j} = 0 \\ ik_k \tilde{C}_T^{jk} = 0 \\ \delta_{jk} \tilde{C}_{TT}^{jk} = 0 \end{cases}$$

$h_{\alpha\beta}$ transforms as a <u>tensor</u> field in <u>Minkowski</u> spacetime

Rotation of spatial axes $\to$ h_{00} : scalar (1 dof)
h_{0j} : vector (3 dof)
h_{jk} : tensor (6 dof)

$$\begin{cases} h_{00} = \dfrac{2U}{c^2} \\ h_{0j} = -\dfrac{4}{c^3} U_j - \dfrac{1}{c} \partial_j A \\ h_{jk} = \dfrac{2}{c^2} V \delta_{jk} + \left(\partial_j \partial_k - \dfrac{1}{3} \delta_{jk} \Delta \right) B + \dfrac{2}{c^2} \partial_{(j} B_{k)} + h_{jk}^{TT} \end{cases}$$

Newtonian grav. pot. in Newt. limit

with
$$\begin{cases} \partial_j U^j = 0 \\ \partial_j B^j = 0 \\ \partial_k h_{TT}^{jk} = 0 \\ \delta_{jk} h_{TT}^{jk} = 0 \end{cases}$$

Gauge transformation

Generator ξ_α decomposed into irreducible pieces :

$$\begin{cases} \xi_0 = \dfrac{\alpha}{c} \\ \xi_j = \dfrac{4}{c^2} \beta_j + \partial_j \gamma \quad \text{where} \quad \partial^j \beta_j = 0 \end{cases}$$

$$h'_{\alpha\beta} = h_{\alpha\beta} - 2\,\partial_{(\alpha}\xi_{\beta)}$$

$$\begin{cases} U' = U - \partial_t \alpha \\ U'_j = U_j + \partial_t \beta_j \\ V' = V - \frac{1}{3} c^2 \Delta \gamma \\ h'^{TT}_{jk} = h^{TT}_{jk} \quad \leftarrow \text{gauge invariant} \\ A' = A + \alpha + \partial_t \gamma \\ B' = B - 2\gamma \\ B'_j = B_j - 4\beta_j \end{cases}$$

Gauge invariants potentials

$\boxed{\text{exo}}$

$$\begin{cases} \Phi \equiv U + \partial_t A + \frac{1}{2} \partial_t^2 B \\ \Phi_j \equiv U_j + \frac{1}{4} \partial_t B_j \\ \Psi = V - \frac{1}{6} c^2 \Delta B \end{cases} \quad \text{and} \quad h^{TT}_{jk}$$

encode information on the gravitational field only, irrespective of the coordinate system

6 d.o.f

Field equations :

Substitute Coulomb-gauge form of the metric perturbation,

$$\begin{cases} h_{00} = \frac{2}{c^2} U \\ h_{0j} = -\frac{4}{c^3} U_j \\ h_{jk} = \frac{2}{c^2} V \delta_{jk} + h_{jk}^{TT} \end{cases}$$

into the general expression for the Riemann tensor in terms of $h_{\alpha\beta}$,

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\beta \partial_\gamma h_{\alpha\delta} - \partial_\beta \partial_\delta h_{\alpha\gamma} - \partial_\alpha \partial_\gamma h_{\beta\delta} + \partial_\alpha \partial_\delta h_{\beta\gamma} \right).$$

Since the Riemann tensor is gauge-invariant, we may evaluate it in any gauge. In the Coulomb gauge,

$$\begin{cases} U = \Phi \\ U_j = \Phi_j \\ V = \Psi \end{cases}$$

are numerically equal to the gauge-invariant potentials.

In the end we obtain expression for the components of $R_{\alpha\beta\gamma\delta}$, in terms of $(\Phi, \Phi_j, \Psi, h_{jk}^{TT})$, valid in <u>any</u> gauge

For the Einstein tensor, this gives

$$\begin{cases} G_{00} = -\dfrac{2}{c^2}\Delta\Psi \\[2mm] G_{0j} = -\dfrac{2}{c^3}\partial_t\partial_j\Psi + \dfrac{2}{c^3}\Delta\Phi_j \\[2mm] G_{jk} = -\dfrac{2}{3c^2}\delta_{jk}\Delta(\Phi-\Psi) - \dfrac{2}{c^4}\delta_{jk}\partial_t^2\Psi - \dfrac{1}{2}\Box h_{jk}^{TT} \\[2mm] \qquad + \dfrac{1}{c^2}\left(\partial_j\partial_k - \dfrac{1}{3}\delta_{jk}\Delta\right)(\Phi-\Psi) + \dfrac{4}{c^4}(\partial_t\partial_{(j}\Phi_{k)}) \end{cases}$$

Similarly, for the energy-momentum tensor, we write

mass density as measured by an observer at rest with respect to the coordinate frame (x^α)

⚠ not proper mass density

$$\begin{cases} T^{00} = \rho c^2 \\[2mm] T^{0j} = (s^j + \partial^j s)\,c \\[2mm] T^{jk} = \tau\,\delta^{jk} + \left(\partial^j\partial^k - \dfrac{1}{3}\delta^{jk}\Delta\right)\sigma + 2\partial^{(j}\sigma^{k)} + \sigma^{jk} \end{cases}$$

momentum density

stress

where
$$\begin{cases} \partial_j s^j = 0 \\ \partial_j \sigma^j = 0 \\ \partial_k \sigma^{jk} = 0 \\ \delta_{jk}\sigma^{jk} = 0 \end{cases}$$

10

10 dof but not all independent because of $\partial_\beta T^{\alpha\beta} = 0$

which implies

$$
\begin{cases}
\Delta s = -\partial_t \rho \\
\Delta \sigma^j = -\partial_t s^j \\
\Delta \sigma = -\frac{3}{2}(\partial_t s + \tau)
\end{cases}
$$

6 free dof $(\rho, s^j, \tau, \sigma^{jk}) \longrightarrow$ 4 determined variables (s, σ^j, σ)

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

(Φ, Ψ) analogous to Bardeen potentials in theory of linear cosmological perturbation

$$\Downarrow$$

$$
\boxed{
\begin{cases}
\Delta \Psi = -4\pi G \rho \\
\Delta(\Phi - \Psi) = -\frac{12\pi G}{c^2}(\partial_t s + \tau) \\
\Delta \Phi_j = -4\pi G s_j \\
\Box h_{jk}^{TT} = -\frac{16\pi G}{c^4}\sigma_{jk}
\end{cases}
}
$$

elliptic equations

hyperbolic wave equation

$\hookrightarrow$ <u>two radiative</u> degrees of freedom out of six

Newtonian limit:

$$\frac{v}{c} \ll 1 \implies \left|\frac{T^{0j}}{T^{00}}\right| \sim \frac{v}{c} \quad \text{and} \quad \left|\frac{T^{jk}}{T^{00}}\right| \sim \left(\frac{v}{c}\right)^2$$

$$\implies |T^{jk}| \ll |T^{0j}| \ll |T^{00}|$$

$$\implies \text{keep only } T^{00} = \rho c^2$$

$$\implies s^j = s = \tau = \sigma = \sigma^j = \sigma^{jk} = 0$$

$$\implies \begin{cases} \Phi = \Psi = U \quad (\text{Coulomb gauge}) \\ \Phi_j = 0 \end{cases}$$

and $\boxed{\Delta U = -4\pi G\rho}$ (Poisson's equation)

We now want to compute the leading-order contribution to the spatial components of the metric perturbation for a source whose internal motion are slow compared to the speed of light (post-newtonian source). We will then compute the TT piece of the metric perturbation to obtain the quadrupole formula for the emitted radiation.

Green's function

$$\begin{cases} \Box \bar{h}_{\alpha\beta} = -\dfrac{16\pi G}{c^4} T_{\alpha\beta} & \text{(wave equation)} \\ \partial^\beta \bar{h}_{\alpha\beta} = 0 & \text{(Lorenz gauge condition)} \end{cases}$$

$$\Box f(t, \vec{x}) = S(t, \vec{x})$$

$$\Box G(t, \vec{x}; t', \vec{x}') = \delta(t-t')\,\delta(\vec{x}-\vec{x}')$$

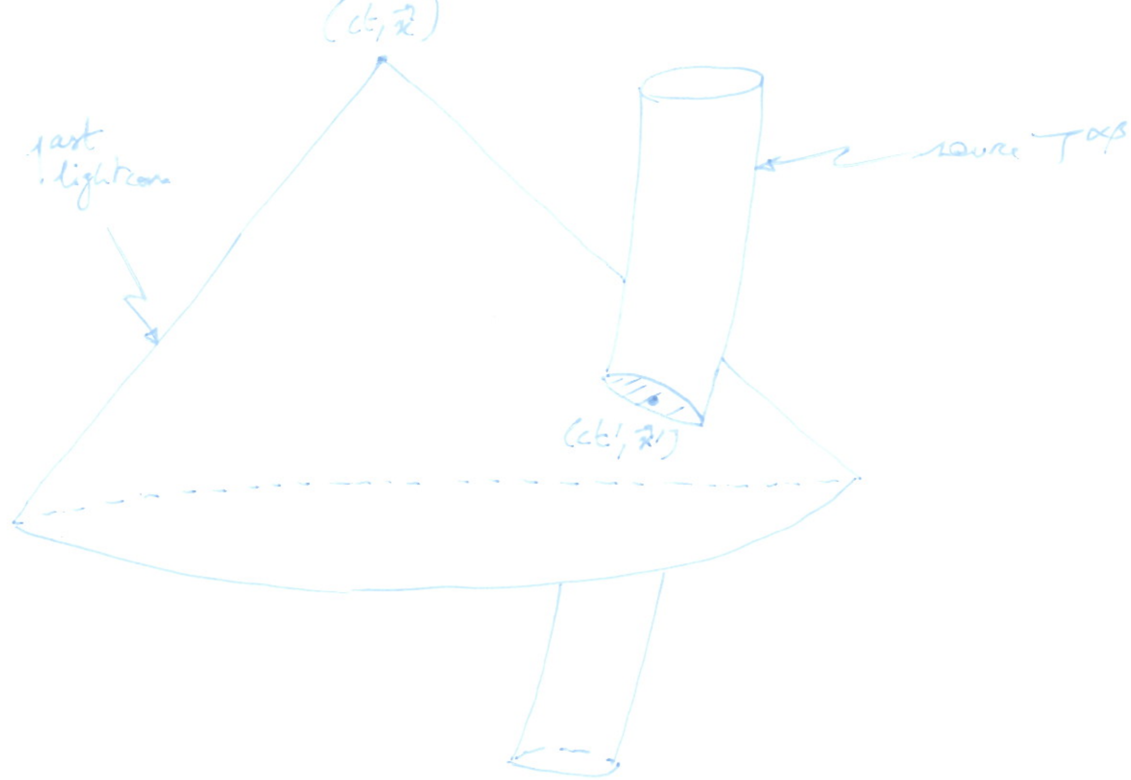
Green's function

$$f(t, \vec{x}) = \int dt'\, d\vec{x}'\, G(t, \vec{x}; t', \vec{x}')\, S(t', \vec{x}')$$

$$G(t, \vec{x}; t', \vec{x}') = -\frac{\delta\Big(t' - \overbrace{\big(t - \tfrac{|\vec{x}-\vec{x}'|}{c}\big)}^{\text{retarded time}}\Big)}{4\pi\,|\vec{x}-\vec{x}'|}$$

retarded Green's function in flat spacetime

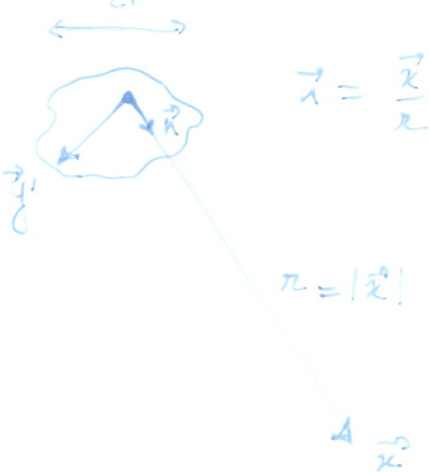
$$\bar{h}_{\alpha\beta}(t,\vec{x}) = \frac{4G}{c^4}\int d^3\vec{x}' \, \frac{T_{\alpha\beta}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|}$$



Change of notation and raise indices using $\eta^{\alpha\beta}$:

$$\boxed{\bar{h}^{\alpha\beta}(t,\vec{x}) = \frac{4G}{c^4}\int_{\text{source}} d^3\vec{y} \, \frac{T^{\alpha\beta}\left(t - \frac{|\vec{x}-\vec{y}|}{c}, \vec{y}\right)}{|\vec{x}-\vec{y}|}}$$

Far-field expansion



$$|\vec{x}-\vec{y}|^2 = r^2 - 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}$$

$$= r^2\Big[1 - \underbrace{\frac{2\vec{y}\cdot\vec{n}}{r}}_{\mathcal{O}\left(\frac{d}{r}\right)} + \mathcal{O}\Big(\frac{d^2}{r^2}\Big)\Big]$$

$$\left.\begin{array}{l} d \sim 10^3\,km \\ r \sim 10\,Mpc \end{array}\right\} \rightarrow \frac{d}{r} \sim 10^{-17}$$

$$|\vec{x}-\vec{y}| = r - \vec{y}\cdot\vec{n} + \mathcal{O}\Big(\frac{1}{r}\Big)$$

$$\boxed{\bar{h}^{\alpha\beta}(t,\vec{x}) = \frac{4G}{c^4 r}\int_{source} d^3y\, T^{\alpha\beta}\Big(\underbrace{t-\frac{r}{c}}_{\equiv t'} + \frac{\vec{n}\cdot\vec{y}}{c}, \vec{y}\Big)}$$

Slow-motion source

Characteristic size d

Characteristic timescale τ

Characteristic internal velocity $v \sim d/\tau$

We assume $\underline{v \ll c}$

$\Updownarrow$

$\underline{d \ll c\tau}$

$$T^{\alpha\beta}\Big(t'+\frac{\vec{n}\cdot\vec{y}}{c}, \vec{y}\Big) = \underbrace{T^{\alpha\beta}(t',\vec{y})}_{\mathcal{O}(1)} + \underbrace{n_j y^j\, \partial_0 T^{\alpha\beta}(t',\vec{y})}_{\mathcal{O}\left(\frac{v}{c}\right)} + \underbrace{n_j n_k y^j y^k\, \partial_0^2 T^{\alpha\beta}(t',\vec{y})}_{\mathcal{O}\left(\frac{v^2}{c^2}\right)}$$

$\sim d$ $\quad \sim T^{\alpha\beta}/c\tau$

$$+\, n_j n_k n_l\, y^j y^k y^l\, \partial_0^3 T^{\alpha\beta}(t',\vec{y}) + \dots$$

$$\bar{h}^{\alpha\beta}(t,\vec{x}) = \frac{4G}{c^4 r}\Big[\int d^3\vec{y}\, T^{\alpha\beta}(t',\vec{y}) + \frac{n_j}{c}\frac{d}{dt}\int d^3\vec{y}\, y^j T^{\alpha\beta}(t',\vec{y})$$
$$+ \frac{n_j n_k}{c^2}\frac{d^2}{dt^2}\int d^3\vec{y}\, y^j y^k T^{\alpha\beta}(t',\vec{y})$$
$$+ \frac{n_j n_k n_l}{c^3}\frac{d^3}{dt^3}\int d^3\vec{y}\, y^j y^k y^l T^{\alpha\beta}(t',\vec{y}) + \cdots\Big]$$

Let's give names to some moments of the components of the energy-momentum tensor:

$$\begin{cases} M(t') \equiv \frac{1}{c^2}\int d^3\vec{y}\, T^{00}(t',\vec{y}) \\ M^j(t') \equiv \frac{1}{c^2}\int d^3\vec{y}\, y^j T^{00}(t',\vec{y}) \\ M^{jk}(t') \equiv \frac{1}{c^2}\int d^3\vec{y}\, y^j y^k T^{00}(t',\vec{y}) \\ M^{jkl}(t') \equiv \frac{1}{c^2}\int d^3\vec{y}\, y^j y^k y^l T^{00}(t',\vec{y}) \end{cases}$$

<u>mass</u>

<u>momentum</u>

$$\begin{cases} P^l(t') \equiv \frac{1}{c}\int d^3\vec{y}\, T^{0l}(t',\vec{y}) \\ P^{lj}(t') \equiv \frac{1}{c}\int d^3\vec{y}\, y^j T^{0l}(t',\vec{y}) \\ P^{ljk}(t') \equiv \frac{1}{c}\int d^3y\, y^j y^k T^{0l}(t',\vec{y}) \end{cases}$$

$$\begin{cases} S^{lm}(t') \equiv \int d^3\vec{y}\, T^{lm}(t',\vec{y}) \\ S^{lmj}(t') \equiv \int d^3\vec{y}\, y^j T^{lm}(t',\vec{y}) \end{cases}$$

<u>stress</u>

Conservation of energy-momentum implies

$$\begin{cases} \dot{M} = 0 \longrightarrow \boxed{M = cst} \\ \dot{M}^k = P^k \\ \dot{M}^{jk} = P^{jk} + P^{kj} \\ \dot{M}^{jkl} = P^{jkl} + P^{klj} + P^{ljk} \end{cases}
\qquad
\begin{cases} \dot{P}^j = 0 \longrightarrow \boxed{P^j = cst} \\ \dot{P}^{jk} = S^{jk} \\ \dot{P}^{jkl} = S^{jkl} + S^{jlk} \end{cases}$$

Indeed, $\partial_\beta T^{\alpha\beta} = 0 \iff \begin{cases} \partial_0 T^{00} + \partial_i T^{0i} = 0 \\ \partial_0 T^{i0} + \partial_j T^{ij} = 0 \end{cases}$

For example, $\dot{M} = \frac{1}{c^2}\frac{d}{dt}\int d^3\vec{y}\, T^{00} = \frac{1}{c}\int d^3\vec{y}\, \partial_0 T^{00} = -\frac{1}{c}\int d^3\vec{y}\, \partial_i T^{0i}$

$$\hookrightarrow \quad \dot{M} = -\frac{1}{c}\oint dS_i\, T^{0i} = 0$$

$$\dot{P}^j = \frac{1}{c}\frac{d}{dt}\int d^3\vec{y}\, T^{0j} = \int d^3\vec{y}\, \partial_0 T^{0j} = -\int d^3\vec{y}\, \partial_k T^{kj}$$

$$\hookrightarrow \quad \dot{P}^j = -\oint dS_k\, T^{kj} = 0$$

$$\dot{M}^k = \frac{1}{c^2}\frac{d}{dt}\int d^3\vec{y}\, y^k T^{00} = \frac{1}{c}\int d^3\vec{y}\, y^k \partial_0 T^{00} = -\frac{1}{c}\int d^3\vec{y}\, y^k \partial_j T^{0j}$$

$$\hookrightarrow \quad \dot{M}^k = -\frac{1}{c}\int d^3\vec{y}\, \cancel{\partial_j (y^k T^{0j})} + \frac{1}{c}\int d^3\vec{y}\, T^{0j} \underbrace{\partial_j y^k}_{=\delta^k_j} = P^k$$

By recursion, we find

$$\ddot{M}^{jk} = \dot{P}^{jk} + \dot{P}^{kj} = S^{jk} + S^{kj}$$

$$\hookrightarrow \quad \boxed{\ddot{M}^{jk} = 2 S^{jk}}$$

Also, $\boxed{\dddot{M}^{jkl} = 6\, \dot{S}^{(jkl)}}$

$$\begin{cases} \bar{h}^{00}(t,\vec{x}) = \dfrac{4G}{c^2 r}\left[M + \dot{M}^j \dfrac{n_j}{c} + \ddot{M}^{jk}(t') \dfrac{n_j n_k}{c^2} + \dddot{M}^{jkl}(t') \dfrac{n_j n_k n_l}{c^3} + \dots \right] \\ \bar{h}^{0j}(t,\vec{x}) = \dfrac{4G}{c^3 r}\left[P^j + \dot{P}^{jk}(t') \dfrac{n_k}{c} + \ddot{P}^{jkl}(t') \dfrac{n_k n_l}{c^2} + \dots \right] \\ \bar{h}^{jk}(t,\vec{x}) = \dfrac{4G}{c^4 r}\left[S^{jk}(t') + \dot{S}^{jkl}(t') \dfrac{n_l}{c} + \dots \right] \end{cases}$$

$$\begin{cases} \bar{h}^{00}(t,\vec{x}) = \dfrac{4G}{c^2 r}\left[M + P^j \dfrac{n_j}{c} + 2 S^{jk}(t') \dfrac{n_j n_k}{c^2} + 6 \dot{S}^{jkl}(t') \dfrac{n_j n_k n_l}{c^3} + \dots \right] \\ \bar{h}^{0j}(t,\vec{x}) = \dfrac{4G}{c^3 r}\left[P^j + S^{jk}(t') \dfrac{n_k}{c} + 2 \dot{S}^{jkl}(t') \dfrac{n_k n_l}{c^2} + \dots \right] \\ \bar{h}^{jk}(t,\vec{x}) = \dfrac{4G}{c^4 r}\left[S^{jk}(t') + \dot{S}^{jkl}(t') \dfrac{n_l}{c} + \dots \right] \end{cases}$$

Gauge transformation

$$h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} - 2\partial_{(\alpha}\xi_{\beta)}$$

$$\bar{h}^{\alpha\beta} \to \bar{h}^{\alpha\beta\,\prime} = \bar{h}^{\alpha\beta} + 2\partial^{(\alpha}\xi^{\beta)} - \eta^{\alpha\beta}\partial_\mu \xi^\mu$$

$$\begin{cases} \delta\bar{h}^{00} = -\partial_0 \xi^0 + \partial_j \xi^j \\ \delta\bar{h}^{0j} = \partial_j \xi^0 - \partial_0 \xi^j \\ \delta\bar{h}^{jk} = \partial_k \xi^j + \partial_j \xi^k - \delta^{jk}\left(\partial_0 \xi^0 + \partial_i \xi^i \right) \end{cases}$$

Far-field expansion $\rightarrow$ leading order in $1/r$

$$\rightarrow \quad \partial_i \left(\frac{1}{r} F(\tau) \right) = \frac{\dot{F}(\tau)}{r} \underbrace{\frac{\partial \tau}{\partial x^i}}_{-\frac{n_i}{c}} + O\left(\frac{1}{r^2}\right)$$

$$\tau = t - \frac{r}{c}$$

Transverse - Traceless gauge

Harmonic gauge non unique

Further gauge transfo. with generator

$$\begin{cases} \xi^0 = \dfrac{G}{c^3 r} \left(P^k{}_k + P^{jk} n_j n_k + S^\ell{}_{\ell k} \dfrac{n^k}{c} + S^{ijk} \dfrac{n_i n_j n_k}{c} \right) \\ \xi^i = \dfrac{G}{c^2 r} \left(4 M^i + 4 P^{ij} \dfrac{n_j}{c} - P^k{}_k \dfrac{n^i}{c} - P^{jk} n_j n_k \dfrac{n^i}{c} \right. \\ \qquad \left. + 4 S^{ijk} \dfrac{n_j n_k}{c^2} - S^\ell{}_{\ell k} \dfrac{n^k n^i}{c^2} - S^{j\ell k} \dfrac{n_j n_\ell n_k\, n^i}{c^2} \right) \end{cases}$$

$\downarrow$ dipole eliminated : COM frame

$$\begin{cases} \bar{h}^{00}_{TT} = \dfrac{4GM}{c^2 r} \\ \bar{h}^{0i}_{TT} = 0 \\ \bar{h}^{ij}_{TT} = \dfrac{4G}{c^4 r} S^{ij}_{TT}(\tau, \vec{n}) \end{cases}$$

harmonic gauge condition satisfied : $\underline{\underline{\partial_\beta \bar{h}^{\alpha\beta}_{TT} = 0}}$

$\leftarrow$ transverse - traceless part of S^{ij}

Extraction of the TT part

Transverse projector $\boxed{P^i{}_j \equiv \delta^i{}_j - n^i n_j}$

$\hookrightarrow$ removes longitudinal component of vectors and tensors

ex: $\quad A^i = \underbrace{A\, n^i}_{\text{longitudinal}} + \underbrace{A_T^i}_{\text{transverse}} \quad$ with $\quad n_i A_T^i = 0$

$$\hookrightarrow \quad P^i{}_j A^j = A^i - n^i n_j A^j$$
$$= \cancel{A n^i} + A_T^i - \cancel{n^i A \overbrace{n_j n^j}^{=1}} - \cancel{n^i n_j A_T^j}$$
$$= A_T^i$$

The transverse projector satisfies

$$\begin{cases} P^i{}_j n^j = 0 \\ P^i{}_i = 2 \\ P^i{}_j P^j{}_k = P^i{}_k \end{cases}$$

~~For any symmetric tensor A^{ij}~~, we define the TT projector as follows:

$\cancel{A^{ij}_{TT} = (TT)^{ij}{}_{kl} A^{kl}}$ ~~where~~ $\quad \boxed{(TT)^{ij}{}_{kl} \equiv P^i{}_k P^j{}_l - \frac{1}{2} P^{ij} P_{kl}}$

$$\begin{cases} (TT)^{ij}{}_{kl}\, n^l = 0 \\ (TT)^{ij}{}_{kl}\, \delta^{kl} = 0 \\ (TT)^{ij}{}_{kl}\, A^{kl}_{TT} = A^{ij}_{TT} \end{cases}$$

For a generic symmetric tensor A^{ij}, decomposed as

$$A^{ij} = \frac{1}{3}\delta^{ij} A + \left(n^i n^j - \frac{1}{3}\delta^{ij}\right) B + 2\, n^{(i} A_T^{j)} + A^{ij}_{TT}$$

with
$$\begin{cases} n_i A^i_T = 0 \\ n_i A^{ij}_{TT} = 0 \\ \delta_{ij} A^{ij}_{TT} = 0 \end{cases}$$

we have $\boxed{(TT)^{ij}{}_{kl}\, A^{kl} = A^{ij}_{TT}}$

Finally, recalling that $S^{ij} = \frac{1}{2}\ddot{M}^{ij}$, we have

$$\boxed{\bar{h}^{ij}_{TT}(t,\vec{x}) = \frac{2G}{c^4 r}\ddot{M}^{ij}_{TT}(\tau,\vec{n}) = \frac{2G}{c^4 r}(TT)^{ij}{}_{kl}(\vec{n})\,\ddot{M}^{kl}(\tau)}$$

The time-independant part of $\bar{h}^{\alpha\beta}_{TT}$ is :

- purely spatial : $\bar{h}^{ij}_{TT}$
- transverse : $\partial_i \bar{h}^{ij}_{TT} \propto n_i \bar{h}^{ij}_{TT} = 0$
- traceless : $\delta_{ij} \bar{h}^{ij}_{TT} = 0$

discuss $\bar{h}_{TT} \equiv - \bar{h}^{00}_{TT} + \cancel{\delta_{ij} \bar{h}^{ij}_{TT}}$
$\neq 0$
but static
$\hookrightarrow \bar{h}_{TT} = 0$ for GW
$\hookrightarrow \bar{h}^{\alpha\beta}_{TT} = h^{\alpha\beta}_{TT}$

Einstein quadrupole formula

$$M^{ij}(t) = \frac{1}{c^2} \int d^3\vec{y}\; y^i y^j \, T^{00}(t,\vec{y}) \quad \text{where} \quad \frac{T^{00}}{c^2} = \underbrace{\rho}_{\text{rest mass density}} + \underbrace{O\left(\frac{1}{c^2}\right)}_{\text{kinetic energy density, internal energy density}}$$

$$M^{ij}(t) = I^{ij}(t) + O\left(\frac{1}{c^2}\right) \quad \text{where} \quad \underline{\underline{I^{ij}(t) \equiv \int_{\text{source}} d^3\vec{y}\; y^i y^j \rho(t,\vec{y})}}$$

Since $\delta^{kl}(\perp\perp)^{ij}{}_{kl} = 0$ we may replace $I^{ij}(t)$ by the quadrupole moment

$$\underline{\underline{Q^{ij}(t) \equiv \int_{\text{source}} d^3\vec{y} \left(y^i y^j - \frac{1}{3} |\vec{y}|^2 \delta^{ij} \right) \rho(t,\vec{y})}}$$

$$\downarrow$$

$$U = -\frac{GM}{r} - \frac{3G}{2r^3} Q^{ij} n_i n_j + \cdots$$

$$\boxed{\bar{h}^{ij}_{TT}(t,\vec{x}) = \frac{2G}{c^4 r} \ddot{Q}^{ij}_{TT}(\tau,\vec{n}) = \frac{2G}{c^4 r} (TT)^{ij}{}_{kl}(\vec{n})\, \ddot{Q}^{kl}(\tau)}$$

Einstein quadrupole formula

Order of magnitude

asymmetry fudge factor
$0 \leqslant s \lesssim 1$

Mass M
Typical size d → Typical quad. moment $Q \sim s M d^2$
Typical frequency ω → Typical velocity $v \sim \omega d$

$$h_0 \sim \frac{2G}{c^4 r}\,\omega^2 s M d^2 \sim \frac{d}{r}\,\frac{R_S}{d}\left(\frac{v}{c}\right)^2 s$$

$R_S = 2GM/c^2$

Most favorable case:

- non spherical $s \lesssim 1$
- compact source $d \gtrsim R_S$
- relativistic speeds $v \lesssim c$

$$h_0 \lesssim \frac{d}{r} \lesssim \frac{GM}{c^2 r}$$

$M = 3 M_\odot$
$r = 40\ \mathrm{Mpc}$ → $\underline{\underline{h_0 \lesssim 10^{-21}}}$

Limit of the linearized theory

From the linearized Einstein equation we derived

$$\bar{h}^{ij}_{TT}(t,\vec{x}) = \frac{2G}{c^4 r}\ddot{I}^{ij}_{TT}(\tau,\vec{x}) = \mathcal{O}(G)$$

where $I^{ij}(t) = \int_{\text{source}} d^3\vec{x}\,\rho(t,\vec{x})\,x^i x^j$

Let's evaluate $\ddot{I}^{ij}$.

$$\frac{d}{dt}\int d^3\vec{x}\,\rho(t,\vec{x})\,f(t,\vec{x}) = \int d^3\vec{x}\left(\rho\frac{\partial f}{\partial t} + f\frac{\partial\rho}{\partial t}\right)$$

$$= \int d^3\vec{x}\left(\rho\frac{\partial f}{\partial t} - f\,\vec{\nabla}\cdot(\rho\vec{v})\right)$$

$$= \int d^3\vec{x}\left(\rho\frac{\partial f}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}f\right) - \cancel{\oint \rho f\,\vec{v}\cdot d\vec{S}}$$

$$= \int d^3\vec{x}\,\rho\frac{df}{dt}$$

For $f(t,\vec{x}) = x^i x^j$ this gives

$$\dot{I}^{ij}(t) = \int d^3\vec{x}\,\rho\,\underbrace{\frac{d}{dt}(x^i x^j)}$$

$$= \cancel{\partial_t(x^i x^j)} + v^k\partial_k(x^i x^j)$$

$$= 2v^k\partial_k x^{(i}x^{j)}$$

$$= 2v^{(i}x^{j)}$$

$$\dot{I}^{ij} = 2\int d^3\vec{x}\,\rho\, v^{(i}x^{j)}$$

$$\ddot{I}^{ij} = 2\int d^3\vec{x}\,\rho\, \underbrace{\frac{d}{dt}\left[v^{(i}x^{j)}\right]}$$

$$= \dot{v}^{(i}x^{j)} + v^{(i}\underbrace{\frac{dx^{j)}}{dt}}$$

$$= \cancel{\partial_t x^j} + v^k\partial_k x^j = v^j$$

$$\ddot{I}^{ij} = 2\int d^3\vec{x}\,\rho\, v^i v^j + 2\int d^3\vec{x}\,\rho\,\dot{v}^{(i}x^{j)}$$

Euler equation : $\rho\dot{v}^i = \rho\,\partial^i U - \partial^i p$

$$\frac{1}{2}\ddot{I}^{ij} = \underbrace{\int \rho\, v^i v^j\, d^3\vec{x}}_{2\times \text{ kinetic energy tensor}} + \underbrace{\int \rho\,\partial^{(i}U\, x^{j)}\, d^3\vec{x}}_{\text{gravitational energy tensor}} - \int \partial^{(i}p\, x^{j)}\, d^3\vec{x}$$

$$U(\vec{x}) = G\int d^3\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

$$\partial^i U(\vec{x}) = +\,G\int d^3\vec{y}\,\rho(\vec{y})\,\partial^i\frac{1}{|\vec{x}-\vec{y}|}$$

$$= -\,G\int d^3\vec{y}\,\rho(\vec{y})\,\frac{x^i-y^i}{|\vec{x}-\vec{y}|^3}$$

$$\int \rho\, \partial^i U\, x^j\, d^3\vec{x} = -G \iint \rho(\vec{x})\rho(\vec{y}) \frac{x^i - y^i}{|\vec{x}-\vec{y}|^3} x^j\, d^3\vec{y}\, d^3\vec{x}$$

$$= -\frac{G}{2} \iint \rho(\vec{x})\rho(\vec{y}) \frac{(x^i - y^i)(x^j - y^j)}{|\vec{x}-\vec{y}|^3} d^3\vec{x}\, d^3\vec{y}$$

<u>Tensor virial theorem</u>:

$$\boxed{\frac{1}{2}\ddot{I}^{ij} = 2T^{ij} + \Omega^{ij} + P\delta^{ij}}$$

where

$$\begin{cases} T^{ij} = \frac{1}{2}\int \rho v^i v^j\, d^3\vec{x} & \text{kinetic energy tensor} \\ \Omega^{ij} = -\frac{G}{2}\iint \rho\rho' \frac{(x-x')^i(x-x')^j}{|\vec{x}-\vec{x}'|^3} d^3\vec{x}'\, d^3\vec{x} & \text{gravitational energy tensor} \\ P = \int p\, d^3\vec{x} & \text{integrated pressure} \end{cases}$$

Taking the trace yields the <u>scalar virial theorem</u>

$$\frac{1}{2}\ddot{I} = 2T + \Omega + 3P.$$

For a periodic system, $\langle \ddot{I}^{ij} \rangle = 0$, such that

$$\boxed{2\langle T^{ij}\rangle + \langle \Omega^{ij}\rangle + \langle P\rangle \delta^{ij} = 0}$$

Hence, as an order of magnitude,

$$|\tau^{ij}| \sim |\Omega^{ij}| \sim P$$

$$\underset{\mathcal{O}(G^0)}{\uparrow} \qquad \underset{\mathcal{O}(G)}{\uparrow} \qquad \underset{\mathcal{O}(G^0)}{\uparrow}$$

$$\frac{1}{2}\ddot{I}^{ij} = 2\tau^{ij} + \Omega^{ij} + P\delta^{ij}$$

$\uparrow$ contributes at $\underline{\underline{\mathcal{O}(G^2)}}$ in $\bar{h}^{ij}_{TT}$!

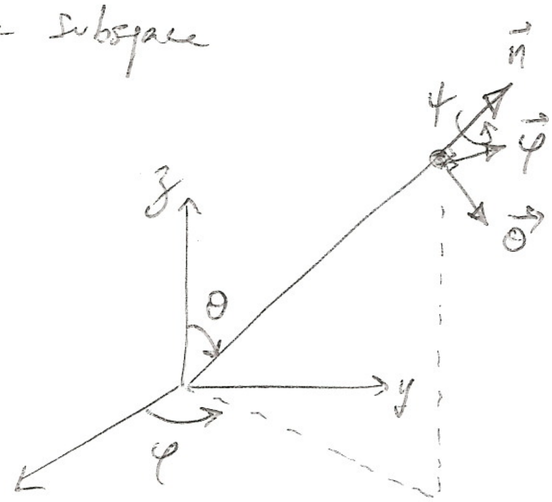
$$\Box \bar{h}^{\alpha\beta} = -\frac{16\pi G}{c^4} T^{\alpha\beta} + \mathcal{O}(h^2)$$

$\uparrow$ $\mathcal{O}(G^2)$ and can't be neglected for gravitationally bound systems

Polarization states

Two degrees of freedom in $\bar{h}^{ij}_{TT} \to (h_+, h_\times)$

We introduce a vectorial $\underbrace{\text{basis}}_{\text{orthonormal}}$ in the transverse subspace



$$\begin{cases} \vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \vec{\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\ \vec{\varphi} = (-\sin\varphi, \cos\varphi, 0) \end{cases}$$

$\vec{n}^2 = \vec{\theta}^2 = \vec{\varphi}^2 = 1$ and $\vec{n}\cdot\vec{\theta} = \vec{n}\cdot\vec{\varphi} = \vec{\theta}\cdot\vec{\varphi} = 0$

$$\delta^{ij} = n^i n^j + \theta^i\theta^j + \varphi^i\varphi^j$$

$$\hookrightarrow \quad P^i{}_j = \theta^i\theta_j + \varphi^i\varphi_j \quad \to \quad (TT)^{ij}{}_{kl} = \cdots$$

polarizations

$$\boxed{\bar{h}^{ij}_{TT} = h_+(\theta^i\theta^j - \varphi^i\varphi^j) + h_\times(\theta^i\varphi^j + \varphi^i\theta^j)}$$

tensorial basis

$$\begin{cases} h_+ = \frac{1}{2}(\theta_i\theta_j - \varphi_i\varphi_j)\,\bar{h}^{ij}_{TT} \\ h_\times = \frac{1}{2}(\theta_i\varphi_j - \varphi_i\theta_j)\,\bar{h}^{ij}_{TT} \end{cases}$$

<u>Note</u>: $\boxed{\dot{\bar{h}}^{ij}_{TT}\,\dot{\bar{h}}^{TT}_{ij} = 2(\dot{h}_+^2 + \dot{h}_\times^2)}\ \boxed{> 0}$

Wave propagating in the z-direction : $\theta = 0$ and we can choose $\varphi = 0$

$$\begin{cases} \vec{n} = (0,0,1) \\ \vec{\theta} = (1,0,0) \\ \vec{\varphi} = (0,1,0) \end{cases} \qquad \begin{cases} \bar{h}^{xx}_{TT} = -\bar{h}^{yy}_{TT} = h_+ \\ \bar{h}^{xy}_{TT} = \bar{h}^{yx}_{TT} = h_\times \end{cases}$$

$$\boxed{\bar{h}^{ij}_{TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

Transverse basis $(\vec{\theta}, \vec{\varphi})$ not unique

Rotation of angle ψ around $\vec{n}$:

$$\begin{cases} \vec{\theta}' = \cos\psi\,\vec{\theta} + \sin\psi\,\vec{\varphi} \\ \vec{\varphi}' = -\sin\psi\,\vec{\theta} + \cos\psi\,\vec{\varphi} \end{cases} \quad \rightarrow \quad \delta^{ij} = n^i n^j + \theta'^i\theta'^j + \varphi'^i\varphi'^j$$

$$\begin{cases} h'_+ = \cos 2\psi\, h_+ + \sin 2\psi\, h_\times \\ h'_\times = -\sin 2\psi\, h_+ + \cos 2\psi\, h_\times \end{cases}$$

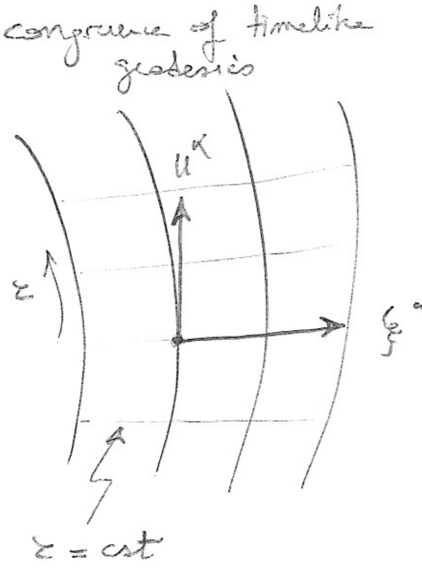
$$\boxed{h'_+ - i h'_\times = e^{2i\psi}(h_+ - i h_\times)}$$

helicity states

helicity $\mathcal{H} = \vec{S}\cdot\vec{n}$

graviton is a spin-2 particle

Geodesic deviation



$$\frac{D^2\xi^\alpha}{d\tau^2} = - R^\alpha{}_{\beta\gamma\delta}\, u^\beta \xi^\gamma u^\delta$$

slow relative velocity

$$\frac{d^2\xi^i}{dt^2} = -c^2 R_{0i0j}\, \xi^j$$

$R_{\alpha\beta\gamma\delta}$ is gauge invariant. In the TT gauge one finds

$$R_{0i0j} = -\frac{1}{2c^2} \ddot{\bar{h}}^{ij}_{TT}$$

$$\frac{d^2\xi^i}{dt^2} = \frac{1}{2} \ddot{\bar{h}}^{ij}_{TT}\, \xi_j$$

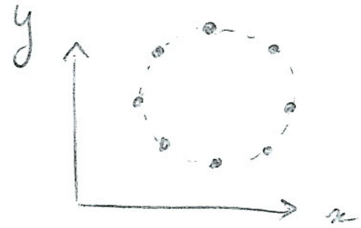
$$\xi^i(t) = \xi^i(0) + \delta\xi^i(t) \quad \text{with} \quad |\delta\vec{\xi}| \ll |\vec{\xi}(0)|$$

$$\delta\ddot{\xi}^i = \frac{1}{2} \ddot{\bar{h}}^{ij}_{TT}\, \xi_j(0)$$

$$\delta\xi^i(t) = \frac{1}{2} \bar{h}^{ij}_{TT}(t)\, \xi_j(0)$$

$$\boxed{\xi^i(t) = \xi^i(0) + \frac{1}{2} \bar{h}^{ij}_{TT}(t)\, \xi_j(0)}$$

Consider an initially circular ring of freely moving particles in an inertial frame, and a GW that travels in the z-direction past the ring, which lies in the x-y plane



$$\bar{h}^{ij}_{TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x(t) = x_0 + \frac{1}{2}\left[h_+(t)\, x_0 + h_\times(t)\, y_0 \right] \\ y(t) = y_0 + \frac{1}{2}\left[h_\times(t)\, x_0 - h_+(t)\, y_0 \right] \\ z(t) = z_0 \end{cases}$$

- Pure h_+ mode:

$$\begin{cases} x(t) = x_0 + \frac{1}{2} h_+(t)\, x_0 & \longrightarrow \quad \frac{\delta x}{x_0} = \frac{1}{2} h_+ \\ y(t) = y_0 - \frac{1}{2} h_+(t)\, y_0 & \longrightarrow \quad \frac{\delta y}{y_0} = -\frac{1}{2} h_+ \end{cases}$$

$$\begin{cases} x = x_0 \left(1 + \frac{1}{2} h_+ \right) \\ y = y_0 \left(1 + \frac{1}{2} h_+ \right) \end{cases}$$

$$\Big(\frac{x}{\underbrace{1+\frac{1}{2}h_+}_{a}}\Big)^2 + \Big(\frac{y}{\underbrace{1-\frac{1}{2}h_+}_{b}}\Big)^2 = x_0^2 + y_0^2$$

Ellipse of eccentricity

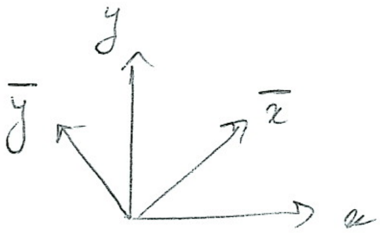
$$e = \sqrt{1-\left(\frac{b}{a}\right)^2} \simeq \sqrt{2h_+} \sim 10^{-10}$$

- Pure h_x mode:

$$\begin{cases} x(t) = x_0 + \frac{1}{2} h_x(t)\, y_0 \\ y(t) = y_0 + \frac{1}{2} h_x(t)\, x_0 \end{cases}$$

$$\frac{1}{2}\left(\frac{x+y}{1+\frac{1}{2}h_x}\right)^2 + \frac{1}{2}\left(\frac{x-y}{1-\frac{1}{2}h_x}\right)^2 = x_0^2 + y_0^2$$ Ellipse rotated by $45°$

$$\begin{cases} \bar{x} = \dfrac{x+y}{\sqrt{2}} \\ \bar{y} = \dfrac{x-y}{\sqrt{2}} \end{cases}$$



Perturbation theory of curved vacuum spacetimes

One parameter family $g_{\alpha\beta}(\varepsilon)$ of solutions of the vacuum Einstein equation

$$g_{\alpha\beta}(\varepsilon) = \overset{\circ}{g}_{\alpha\beta} + \varepsilon\, h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta} + \mathcal{O}(\varepsilon^3) \qquad \text{where } G_{\alpha\beta}[\overset{\circ}{g}] = 0$$

- $\overset{\circ}{g}_{\alpha\beta}$: background $g_{\alpha\beta}(0)$
- $h_{\alpha\beta}$: linear metric pert. $\left.\dfrac{d g_{\alpha\beta}}{d\varepsilon}\right|_{\varepsilon=0}$
- $j_{\alpha\beta}$: second-order metric pert. $\left.\dfrac{1}{2}\dfrac{d^2 g_{\alpha\beta}}{d\varepsilon^2}\right|_{\varepsilon=0}$
- ε: formal expansion parameter

Repeat perturbation theory over Minkowski with $\begin{cases} \eta_{\alpha\beta} \to \overset{\circ}{g}_{\alpha\beta} \\ \partial_\alpha \to \overset{\circ}{\nabla}_\alpha \end{cases}$ $\quad (\overset{\circ}{\nabla}_\alpha \overset{\circ}{g}_{\beta\gamma} = 0)$

$$0 = G_{\alpha\beta} = \cancel{G_{\alpha\beta}[\overset{\circ}{g}]} + \varepsilon\, G^{(1)}_{\alpha\beta}[h;\overset{\circ}{g}] + \varepsilon^2 G^{(1)}_{\alpha\beta}[j;\overset{\circ}{g}] + \varepsilon^2 G^{(2)}_{\alpha\beta}[h;\overset{\circ}{g}] + \mathcal{O}(\varepsilon^3)$$

$G^{(1)}_{\alpha\beta}$: linearized Einstein tensor

- $$G^{(1)}_{\alpha\beta}[h;\overset{\circ}{g}] = -\frac{1}{2}\Box \bar{h}_{\alpha\beta} + \overset{\circ}{R}_{\gamma\beta\alpha\delta}\bar{h}^{\gamma\delta} + \overset{\circ}{\nabla}_{(\alpha}\overset{\circ}{\nabla}_{|\gamma|}\bar{h}^{\gamma}{}_{\beta)} - \frac{1}{2}\overset{\circ}{g}_{\alpha\beta}\overset{\circ}{\nabla}_\gamma\overset{\circ}{\nabla}_\delta\bar{h}^{\gamma\delta}$$
  with $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\overset{\circ}{g}_{\alpha\beta}\underbrace{\overset{\circ}{g}^{\gamma\delta}h_{\gamma\delta}}_{\equiv h}$

- $$G^{(2)}_{\alpha\beta}[h;\overset{\circ}{g}] \sim h_{\alpha\beta}\overset{\circ}{\nabla}_\gamma\overset{\circ}{\nabla}_\delta h_{\rho\sigma} + \overset{\circ}{\nabla}_\alpha h_{\beta\gamma}\overset{\circ}{\nabla}_\delta h_{\rho\sigma}$$

~~9781021400~~

$$\begin{cases} G^{(1)}_{\alpha\beta}[h;\overset{\circ}{g}] = 0 \\ G^{(1)}_{\alpha\beta}[j;\overset{\circ}{g}] = -G^{(2)}_{\alpha\beta}[h;\overset{\circ}{g}] \end{cases}$$

The geometric optics regime, or shortwave approximation

Let's go back to the linearized approximation for a moment:

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \mathcal{O}(\varepsilon^2)$$

In most circumstances, this separation into background and perturbation is a mathematical device, as no unique separation is determined by local physical measurements.

Given a one-parameter family $g_{\alpha\beta}(\varepsilon)$, $\mathring{g}_{\alpha\beta}$ and $h_{\alpha\beta}$ are uniquely specified, but a given physical solution is described by a single physical metric

$$g_{\alpha\beta}(\varepsilon_0) = \mathring{g}_{\alpha\beta} + \varepsilon_0 h_{\alpha\beta} + \mathcal{O}(\varepsilon_0^2),$$

and a local physical measurement is sensitive to the <u>sum</u> of $\mathring{g}_{\alpha\beta}$ and $\varepsilon_0 h_{\alpha\beta}$.

However, in special circumstances, a <u>unique</u> separation into background + perturbation is determined by local physical measurements, and it is only in this context that GWs can be defined. This happens when there is a separation of scales between

λ : characteristic wavelength of the waves

$\mathcal{L}$: characteristic lengthscale of the background curvature

$\boxed{\lambda \ll \mathcal{L}}$: geometric optics regime

Then we can define

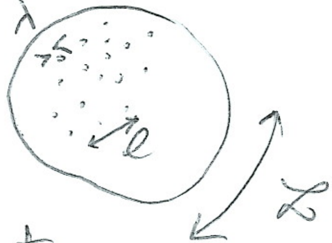
$$\boxed{\begin{aligned} \mathring{g}_{\alpha\beta} &\equiv \langle g_{\alpha\beta} \rangle \\ \varepsilon h_{\alpha\beta} &\equiv g_{\alpha\beta} - \mathring{g}_{\alpha\beta} \end{aligned}}$$

← varies over $\mathcal{L}$, $\partial \mathring{g} \sim \mathring{g}/\mathcal{L}$

← varies over λ, $\partial h \sim \dfrac{h}{\lambda}$

where $\langle \ldots \rangle$ denotes a (covariant) averaging over lengthscales ℓ such that

$$\lambda \ll \ell \ll \mathcal{L} .$$



Such an averaging smoothes out the fast-varying components.

We will show that

$$\varepsilon h_{\alpha\beta} \Longrightarrow T^{\text{eff, GW}}_{\alpha\beta} \sim \frac{\varepsilon^2 h^2}{\lambda^2}$$

that contributes to the curvature of the background metric, which is $\lesssim \dfrac{1}{\mathcal{L}^2}$

Hence $\dfrac{\varepsilon^2 h^2}{\lambda^2} \lesssim \dfrac{1}{\mathcal{L}^2} \iff \varepsilon h \lesssim \dfrac{\lambda}{\mathcal{L}} \ll 1$

⇓

perturbation theory is OK

How does this definition of a GW compare to that introduced earlier, in terms of the TT component of the deviation from the flat Minkowski metric ?

It is more general. But it agrees far away from the source. Indeed, $\bar{h}^{ij}_{TT}$ varies over $\lambda \ll \mathcal{L} = |\vec{x}| \simeq r$, the scale of variation of the other components. Recall that in the TT gauge,

$$\begin{cases} \bar{h}^{00}_{TT} = \dfrac{4GM}{c^2 r} & \leftarrow \text{ part of the background varying over scale } r \\ \bar{h}^{0i}_{TT} = 0 & \\ \bar{h}^{ij}_{TT} = \dfrac{2G}{c^4 r} \ddot{I}^{ij}_{TT}(\tau, \vec{n}) & \leftarrow \text{ GW varying over scale } \lambda \end{cases}$$

$$h^{\alpha\beta}_{TT} = \bar{h}^{\alpha\beta}_{TT} - \frac{1}{2}\eta^{\alpha\beta}\bar{h} \implies g_{\alpha\beta}$$

Schwarzschild metric in harmonic coordinates to linear order in M/r

$$\begin{cases} g_{00} = \boxed{-1 + \dfrac{2GM}{rc^2}} & \mathring{g}_{\alpha\beta} \\ g_{0i} = 0 & \\ g_{ij} = \delta_{ij}\left(1 + \dfrac{2GM}{c^2 r}\right) + \boxed{\dfrac{2G}{c^4 r}\ddot{I}_{TT}(\tau, \vec{n})} & h_{\alpha\beta} \end{cases}$$

$$ds^2 = -\left(\frac{1 - M/r}{1 + M/r}\right) dt^2 + \frac{1 + M/r}{1 - M/r} dr^2 + (r+M)^2 d\Omega^2$$

Effective stress-energy tensor of gravitational waves

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta} + O(\varepsilon^3)$$

$$\begin{cases} G_{\alpha\beta}[\mathring{g}] = 0 \\ G^{(1)}_{\alpha\beta}[h; \mathring{g}] = 0 \\ G^{(1)}_{\alpha\beta}[j; \mathring{g}] = -G^{(2)}_{\alpha\beta}[h; \mathring{g}] \sim h \nabla\nabla h + \nabla h \nabla h \end{cases}$$

The quadratic source creates a slowly varying piece and a fast varying piece in $j_{\alpha\beta}$:

$$j_{\alpha\beta} = \underbrace{\langle j_{\alpha\beta} \rangle}_{\text{varies over } \mathcal{L}} + \underbrace{\Delta j_{\alpha\beta}}_{\text{varies over } \lambda}$$

$$g_{\alpha\beta} = \underbrace{\left(\mathring{g}_{\alpha\beta} + \varepsilon^2 \langle j_{\alpha\beta} \rangle \right)}_{\text{varies over } \mathcal{L}} + \underbrace{\left(\varepsilon h_{\alpha\beta} + \varepsilon^2 \Delta j_{\alpha\beta} \right)}_{\text{varies over } \lambda} + O(\varepsilon^3)$$

<u>Ex</u>: $h = \cos\omega t \Rightarrow j \sim h^2 = \cos^2\omega t = \underset{\substack{\uparrow \\ \langle j \rangle}}{\frac{1}{2}} + \frac{1}{2} \underset{\substack{\uparrow \\ \Delta j}}{\cos 2\omega t}$

Field equations for $\langle j_{\alpha\beta}\rangle$ and $\Delta j_{\alpha\beta}$ separately:

$$\langle G^{(1)}_{\alpha\beta}[j;\mathring{g}]\rangle \underset{\uparrow}{=} G^{(1)}_{\alpha\beta}[\langle j\rangle;\mathring{g}] = -\langle G^{(2)}_{\alpha\beta}[h;\mathring{g}]\rangle$$

commutation of $\langle\ldots\rangle$ and $\mathring{\nabla}$

$G^{(1)}$ is linear

$$G^{(1)}_{\alpha\beta}[\Delta j;\mathring{g}] = G^{(1)}_{\alpha\beta}[j;\mathring{g}] - G^{(1)}_{\alpha\beta}[\langle j\rangle;\mathring{g}]$$

$$= -G^{(2)}_{\alpha\beta}[h;\mathring{g}] + \langle G^{(2)}_{\alpha\beta}[h;\mathring{g}]\rangle$$

Since $G_{\alpha\beta}[\mathring{g}] = 0$ we have the effective Einstein equation for the slowly-varying part of the metric

$$\boxed{\begin{aligned} G_{\alpha\beta}[\mathring{g}+\varepsilon^2\langle j\rangle] &= \frac{8\pi G}{c^4}T^{GW,\,eff}_{\alpha\beta} + \mathcal{O}(\varepsilon^3) \\ \text{where}\quad T^{GW,\,eff}_{\alpha\beta} &= -\frac{c^4}{8\pi G}\langle G^{(2)}_{\alpha\beta}[\varepsilon h,\mathring{g}]\rangle \end{aligned}}$$

All terms vary on lengthscale $\mathcal{L}$

$$\tilde{\nabla}^{\kappa}T^{GW,\,eff}_{\kappa\beta} = 0$$

compatible with $\mathring{g}+\varepsilon^2\langle j\rangle \equiv \tilde{g}$

$$\mathring{\nabla}^{\kappa}T^{GW,\,eff}_{\kappa\beta} = \mathcal{O}(\varepsilon^2)$$

A GW $h_{\alpha\beta}$ gives rise to a correction $\varepsilon^2\langle j_{\alpha\beta}\rangle$ of the background metric $\mathring{g}_{\alpha\beta}$ through an effective energy-momentum tensor $T^{GW,\,eff}_{\alpha\beta}$

Just like any other form of matter source

Schematically,

$$T_{\alpha\beta}^{GW,\mathrm{eff}} \sim \langle h_{\alpha\beta} \mathring{\nabla}_\gamma \mathring{\nabla}_\delta h_{\rho\sigma} \rangle + \langle \mathring{\nabla}_\alpha h_{\rho\sigma} \mathring{\nabla}_\delta h_{\rho\sigma} \rangle$$

An explicit calculation reveals that

$$\boxed{T_{\alpha\beta}^{GW,\mathrm{eff}} = \frac{c^4}{32\pi G} \Big\langle \mathring{\nabla}_\alpha \bar{h}_{\gamma\delta} \mathring{\nabla}_\beta \bar{h}^{\gamma\delta} - \frac{1}{2} \mathring{\nabla}_\alpha \bar{h} \mathring{\nabla}_\beta \overbrace{\bar{h}}^{\mathring{g}^{\alpha\beta} \bar{h}_{\alpha\beta}} - 2 \mathring{\nabla}_{(\alpha} \bar{h}_{\beta)\gamma} \nabla_\delta \bar{h}^{\gamma\delta} \Big\rangle}$$

$(\varepsilon = 1)$

~~any gauge~~

Isaacson's effective stress-energy tensor

where the following "tricks" are used:

- $\underbrace{\mathring{\nabla}_\alpha \mathring{\nabla}_\beta h_{..}}_{\sim \frac{h}{\lambda^2}} = \underbrace{\mathring{\nabla}_\beta \mathring{\nabla}_\alpha h_{..}}_{\sim \frac{h}{\lambda^2}} + \underbrace{\cancel{R_{....} h^{..}}}_{\sim \frac{h}{\mathcal{L}^2} \ll \frac{h}{\lambda^2}} + \cancel{R_{...} h^{..}}$

- $\langle h_\alpha{}^\gamma \mathring{\nabla}_\delta \mathring{\nabla}_\gamma h_\beta{}^\delta \rangle = \cancel{\langle \mathring{\nabla}_\delta (h_\alpha{}^\gamma \mathring{\nabla}_\gamma h_\beta{}^\delta) \rangle}$ $\leftarrow \int dV\, \nabla$
$\qquad - \langle \mathring{\nabla}_\delta h_\alpha{}^\gamma \mathring{\nabla}_\gamma h_\beta{}^\delta \rangle$

One can check that $T_{\alpha\beta}^{GW,\mathrm{eff}}$ is invariant under gauge transformations of the form $h_{\alpha\beta} \to h_{\alpha\beta} - 2 \mathring{\nabla}_{(\alpha} \xi_{\beta)}$

Lorenz gauge

We can always set $\overset{\circ}{\nabla}_\beta \bar{h}^{\alpha\beta} = 0$

In vacuum, we can further impose $\bar{h} = -h = 0$ (see Wald)

$\Longrightarrow \bar{h}_{\alpha\beta} = h_{\alpha\beta}$

$$\boxed{T^{GW,\,eff}_{\alpha\beta} = \frac{c^4}{32\pi G} \langle \overset{\circ}{\nabla}_\alpha h_{\gamma\delta} \overset{\circ}{\nabla}_\beta h^{\gamma\delta} \rangle}$$

valid anywhere in spacetime

Wave zone

For an asymptotically flat spacetime, and far away from any source of radiation (wave zone), the metric perturbation is that of flat spacetime, so that $\overset{\circ}{g}_{\alpha\beta} = \eta_{\alpha\beta}$ and $\overset{\circ}{\nabla}_\alpha = \partial_\alpha$.

$$\boxed{T^{GW,\,eff}_{\alpha\beta} = \frac{c^4}{32\pi G} \langle \partial_\alpha h_{\gamma\delta} \partial_\beta h^{\gamma\delta} \rangle}$$

valid in the wave zone

TT gauge

The radiative degrees of freedom are contained in the TT part of the spatial part of the metric, the only relevant components in the TT gauge.

$$\boxed{T^{GW,\,eff}_{\alpha\beta} = \frac{c^4}{32\pi G} \langle \partial_\alpha h^{TT}_{ij} \partial_\beta h^{ij}_{TT} \rangle}$$

valid in the TT gauge

Important simplification in the wave zone:

$$h^{\alpha\beta} = \frac{1}{r} F^{\alpha\beta}(\tau, \vec{n}) \implies \partial_i h^{\alpha\beta} = -\frac{1}{c} n_i \dot{h}^{\alpha\beta} + O\left(\frac{1}{r^2}\right)$$

<u>Ex</u>: $$h^{jk}_{TT} = \frac{2G}{c^4 r} \ddot{I}^{jk}_{TT}(\tau, \vec{n}) = \frac{2G}{c^4 r} (TT)^{jk}{}_{pq}(\vec{n}) \, \ddot{I}^{pq}(\tau)$$

$$\hookrightarrow \partial_i h^{jk}_{TT} = \frac{2G}{c^4 r} (TT)^{jk}{}_{pq}(\vec{n}) \underbrace{\partial_i \ddot{I}^{pq}(\tau)}_{-\frac{n_i}{c}\dddot{I}^{pq}(\tau)} + O\left(\frac{1}{r^2}\right)$$

because $\partial_i \frac{1}{r} = -\frac{n_i}{r^2}$ and $\partial_i n^j = \frac{\delta^{ij} - n^i n^j}{r}$

<u>Gravitational wave flux of energy</u>

Flux of energy in the direction i given by $\mathcal{F}^i = c\, T^{0i}$

$$T^{0i} = \eta^{0\alpha} \eta^{i\beta} T_{\alpha\beta} = \underbrace{\eta^{00}}_{-1} \underbrace{\eta^{ij}}_{\delta^{ij}} T_{0j} = -T_{0i}$$

where $$T_{0i} = \frac{c^4}{32\pi G} \langle \underbrace{\partial_0 h^{TT}_{jk}}_{=\frac{1}{c}\dot{h}^{TT}_{jk}} \underbrace{\partial_i h^{jk}_{TT}}_{-\frac{1}{c} n_i \dot{h}^{jk}_{TT}} \rangle$$

$$= -\frac{c^2}{32\pi G} n_i \langle \dot{h}^{TT}_{jk} \dot{h}^{jk}_{TT} \rangle$$

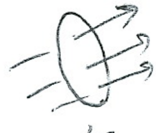
Hence $\boxed{\vec{\mathcal{F}} = \mathcal{F}\vec{n} \quad \text{with} \quad \mathcal{F} = \frac{c^3}{32\pi G} \langle \dot{h}_{jk}^{TT} \dot{h}^{jk}_{TT} \rangle}$

We proved earlier that

$$h^{jk}_{TT} = h_+ (\theta^j\theta^k - \varphi^j\varphi^k) + h_\times (\theta^j\varphi^k + \varphi^j\theta^k)$$

$$\Rightarrow \quad \dot{h}_{jk}^{TT} \dot{h}^{jk}_{TT} = 2(\dot{h}_+^2 + \dot{h}_\times^2) > 0$$

So finally



dS

$$\boxed{\mathcal{F} = \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle} = \frac{d^2E}{dt\,dS}$$

<u>Example</u> : plane wave propagating in the z-direction

$$\begin{cases} h_+ = h_0 \cos(\omega(t - z/c) + \varphi_0) \\ h_\times = 0 \end{cases}$$

$$\mathcal{F} = \frac{c^3}{32\pi G}\omega^2 h_0^2 = \frac{\pi c^3}{8G} f^2 h_0^2 = 1{,}5 \left(\frac{f}{1\,\mathrm{kHz}}\right)^2 \left(\frac{h_0}{10^{-22}}\right)^2 \mathrm{mW/m^2}$$

Despite $h_0 \ll 1$, $\mathcal{F} \sim \mathrm{mW/m^2}$, which is comparable to the flux of reflected sunlight from a full moon! A GW with a tiny amplitude can carry a large amount of energy $\Rightarrow$ spacetime = rigid "medium"

Gravitational-wave luminosity or total radiated power

$$\boxed{\begin{aligned} \mathcal{P} &= \lim_{r\to+\infty} \oint dS_i\, \mathcal{F}^i \qquad \text{where} \begin{cases} dS_i = r^2 d\Omega\, n_i \\ \mathcal{F}^i = \mathcal{F} n^i \end{cases} \\ &= \lim_{r\to+\infty} \frac{c^3 r^2}{16\pi G} \oint d\Omega\, \underbrace{\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle}_{\sim 1/r^2 \text{ because } h^{ij}_{TT} \sim 1/r} \end{aligned}}$$

Energy balance for GWs :

$$\partial_\alpha T^{\alpha\beta}_{GW} = 0 \iff \begin{cases} \partial_0 T^{00}_{GW} + \partial_i T^{i0}_{GW} = 0 \\ \partial_0 T^{0i}_{GW} + \partial_j T^{ji}_{GW} = 0 \end{cases}$$

$$\begin{aligned} \frac{d}{dt} \underbrace{\int d^3x\, T^{00}_{GW}}_{\text{energy in GWs}} &= c\int d^3x\, \partial_0 T^{00}_{GW} \\ &= -c\int d^3x\, \partial_i T^{0i}_{GW} \\ &= -c\oint dS_i\, T^{0i}_{GW} \\ &= -\oint dS_i\, \mathcal{F}^i < 0 \end{aligned}$$

Radiated linear momentum

Similarly, $$\underbrace{\frac{d}{dt}\int d^3x \, \frac{T^{0i}_{GW}}{c}}_{\text{linear momentum in GWs}} = -\oint dS_j \, T^{ij}_{GW}$$

$$T^{ij} = T_{ij} = \frac{c^4}{32\pi G} \langle \partial_i h^{TT}_{kl} \, \partial_j h^{kl}_{TT} \rangle$$

$$= \frac{c^2}{32\pi G} \langle \dot{h}^{TT}_{kl} \, \dot{h}^{kl}_{TT} \rangle n_i n_j$$

$$= \frac{c^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle n_i n_j$$

$\downarrow$

$$\boxed{P_i = \lim_{r \to \infty} \frac{c^2 r^2}{16\pi G} \oint d\Omega \, n_i \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle}$$

Einstein quadrupole formula

$$\mathcal{P} = \lim_{r\to\infty} \frac{c^3 r^2}{32\pi G} \oint d\Omega \, \langle \dot{h}^{TT}_{ij} \dot{h}^{TT}_{ij} \rangle$$

where $\dot{h}^{TT}_{ij}(t,\vec{x}) = \dfrac{2G}{c^4 r} (TT)^{ij}{}_{kl}(\vec{n}) \dddot{Q}^{kl}(\tau)$

$$\downarrow$$

$$\mathcal{P} = \frac{G}{2c^5} \langle\langle \overbrace{(TT)^{ij}{}_{kl} (TT)_{ij}{}^{pq}}^{= (TT)_{kl}{}^{pq}} \rangle\rangle \langle \dddot{Q}^{kl}(\tau) \dddot{Q}_{pq}(\tau) \rangle$$

where $\quad \langle\langle f(\vec{n}) \rangle\rangle \equiv \dfrac{1}{4\pi} \oint d\Omega \, f(\vec{n})$

$$(TT)_{kl}{}^{pq} = \delta^p_k \delta^q_l - \delta^p_k n^q n_l - \delta^q_l n^p n_k + \frac{1}{2} n^p n^q n_k n_l$$
$$+ (\text{terms} \propto \delta^{pq} \text{ or } \delta_{kl})$$

$$\begin{cases} \langle\langle n^q n_l \rangle\rangle = \frac{1}{3} \delta^q_l \\ \langle\langle n^p n_k \rangle\rangle = \frac{1}{3} \delta^p_k \\ \langle\langle n^p n^q n_k n_l \rangle\rangle = \frac{1}{15} (\delta^{pq} \delta_{kl} + \delta^p_k \delta^q_l + \delta^p_l \delta^q_k) \end{cases}$$

$$\langle\langle (TT)_{kl}{}^{pq} \rangle\rangle = \delta^p_k \delta^q_l \underbrace{\left(1 - \frac{2}{3} + \frac{1}{2} \times \frac{2}{15}\right)}_{= 2/5} + (\text{terms} \propto \delta^{pq} \text{ or } \delta^{kl})$$

Finally, $$\mathcal{P} = \frac{G}{\cancel{2}c^5}\,\frac{\cancel{2}}{5}\,\delta^p{}_k\,\delta^q{}_l \left\langle \dddot{Q}^{kl}\,\dddot{Q}_{pq} \right\rangle$$

$$\hookrightarrow \quad \boxed{\mathcal{P} = \frac{G}{5c^5} \left\langle \dddot{Q}^{kl}\,\dddot{Q}_{kl} \right\rangle}$$

Einstein quadrupole formula (1918)

Analogy with the radiated power of a slowly-moving distribution of accelerated charges:

$$\mathcal{P}_{e.m.} = \frac{2}{3}\,\frac{1}{4\pi\varepsilon_0}\,\frac{1}{c^3} \left\langle \ddot{D}^i\,\ddot{D}_i \right\rangle$$

$G \longleftrightarrow \dfrac{1}{4\pi\varepsilon_0}$ Grav: conservation of mass and linear mom.

EM: " " charge

Order of magnitude

$$Q \sim s M d^2 \rightarrow \dddot{Q} \sim s \omega^3 M d^2$$

$$\mathcal{P} \sim \frac{G}{c^5} s^2 \omega^6 M^2 d^4 \sim \frac{c^5}{4G} \left(\frac{2GM}{c^2 d}\right)^2 \left(\frac{v}{c}\right)^6 s^2$$

For $s \lesssim 1$, $v \lesssim c$ and $d \gtrsim \frac{2GM}{c^2}$,

$$\mathcal{P} \lesssim \mathcal{P}_{max} \sim \frac{c^5}{4G} \simeq 10^{52}\,W$$

$\mathcal{P}_{sun} \simeq 3,8 \times 10^{26}\,W$

$\mathcal{P}_{galaxy} \simeq 10^{37}\,W$

$\mathcal{P}_{all\ gal.} \simeq 10^{49}\,W$

GW150912

$\mathcal{P}_{peak} \simeq 3.6 \times 10^{49}\,W$

$\simeq 3 M_\odot c^2 / 250\,ms$

$\simeq 200\, M_\odot c^2 / s$