Celestial mechanics in Kerr spacetime

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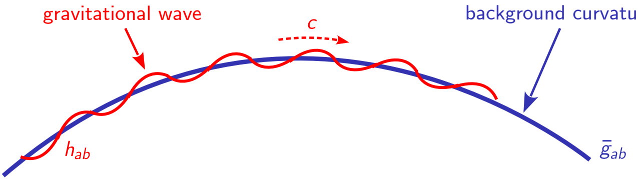
CQG **31** (2014) 097001, arXiv:1311.3836 [gr-qc]
PRL **113** (2014) 161101, arXiv:1404.6133 [gr-qc]
CQG **34** (2017) 134001, arXiv:1612.02504 [gr-qc]

Outline

Outline

What is a gravitational wave ?

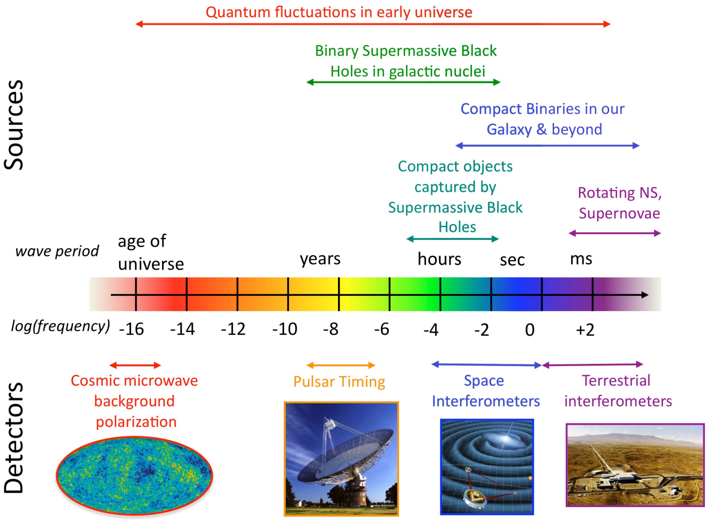
A gravitational wave is a tiny ripple in the curvature of spacetime that propagates at the vacuum speed of light



$$\Box h_{ab} + 2\bar{R}_{abcd}h^{cd} = -16\pi T_{ab}$$

Key prediction of Einstein's general theory of relativity

The gravitational-wave spectrum



Gravitational-wave science

Fundamental physics

- Strong-field tests of GR
- Black hole no-hair theorem
- Cosmic censorship conjecture
- Dark energy equation of state
- Alternatives to general relativity

Astrophysics

- Formation and evolution of compact binaries
- Origin and mechanisms of γ-ray bursts
- Internal structure of neutron stars

Cosmology

- Cosmography and measure of Hubble's constant
- Origin and growth of supermassive black holes
- Phase transitions during primordial Universe

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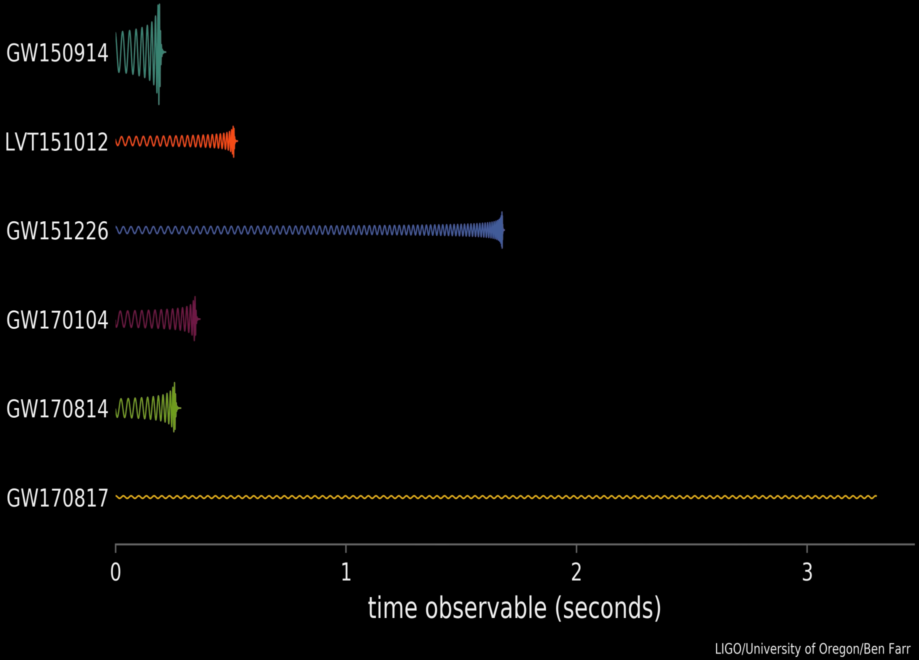
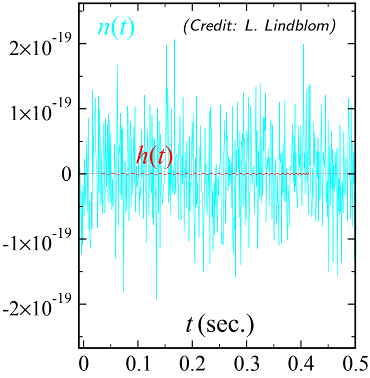
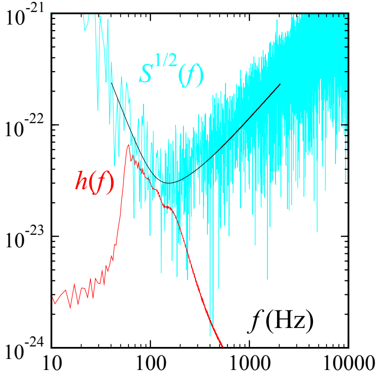
Ground-based interferometric detectors


GW150914
LVT151012
GW151226
GW170104
GW170814
GW170817
0
1
2
3
time observable (seconds)
LIGO/University of Oregon/Ben Farr

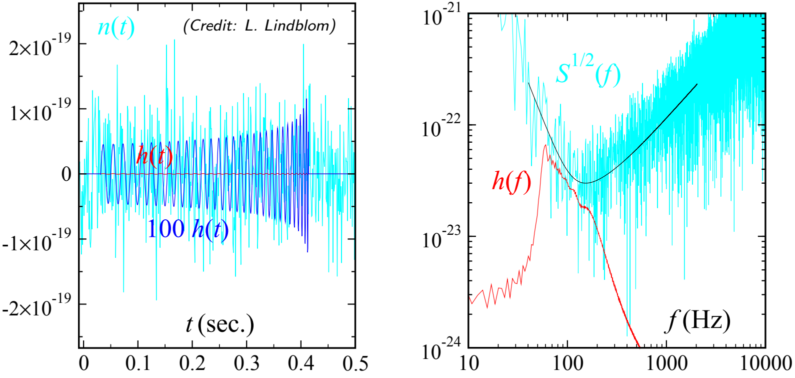
Need for accurate template waveforms





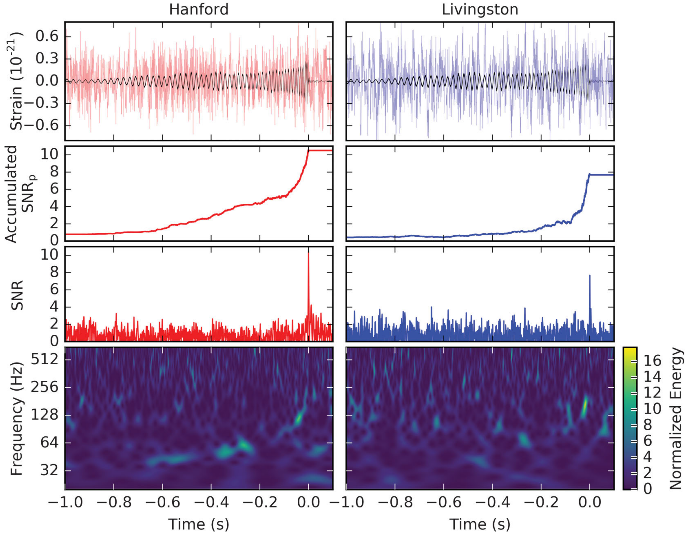
If the expected signal is *known in advance* then $n(t)$ can be filtered and $h(t)$ recovered by matched filtering $\longrightarrow$ **template waveforms**

Need for accurate template waveforms



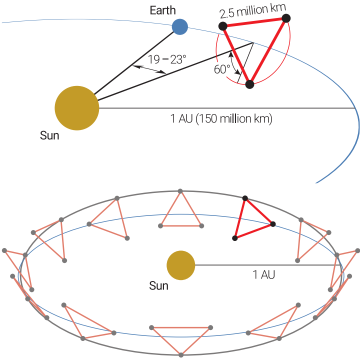
If the expected signal is *known in advance* then $n(t)$ can be filtered and $h(t)$ recovered by matched filtering $\longrightarrow$ **template waveforms**

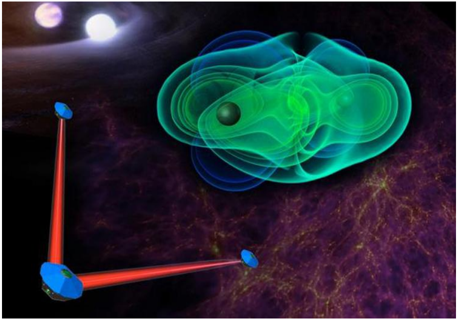
A recent example: the event GW151226



[PRL **116** (2016) 241103]

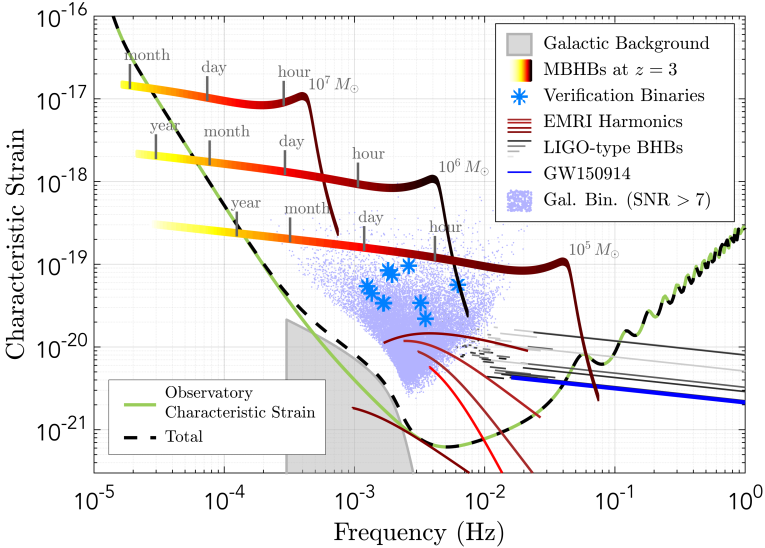
LISA: a gravitational antenna in space





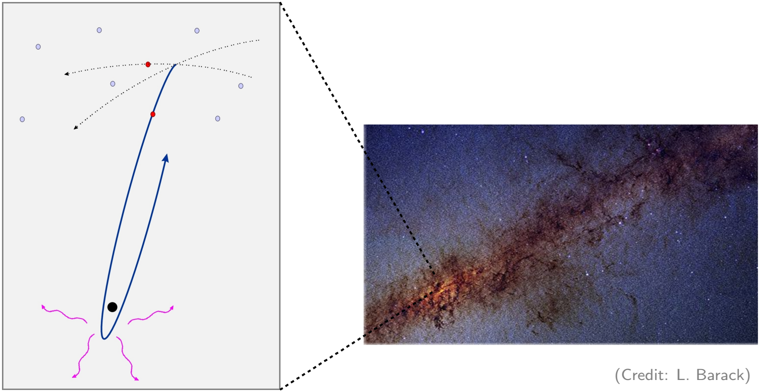
The *LISA mission* proposal was accepted by ESA in 2017 for L3 slot, with a launch planned for 2034 [http://www.lisamission.org]

LISA sources of gravitational waves



Outline

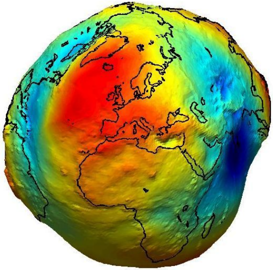
Extreme mass ratio inspirals



(Credit: L. Barack)

- LISA sensitive to $M_{\rm BH} \sim 10^5 - 10^7 M_\odot \to q \sim 10^{-7} - 10^{-4}$
- $T_{\rm orb} \propto M_{\rm BH} \sim$ hr and $T_{\rm insp} \propto M_{\rm BH}/q \sim$ yrs

Geodesy



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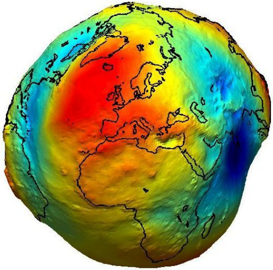

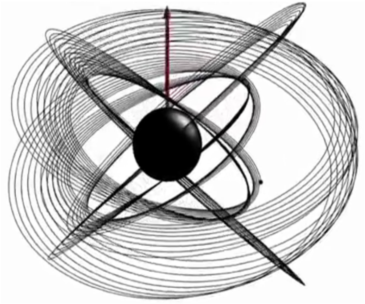

(Credit: S. Drasco)

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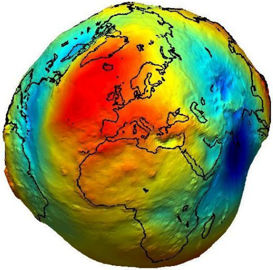
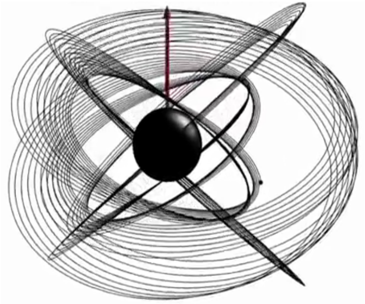


Botriomeladesy



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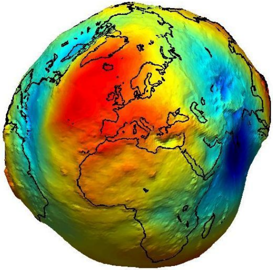

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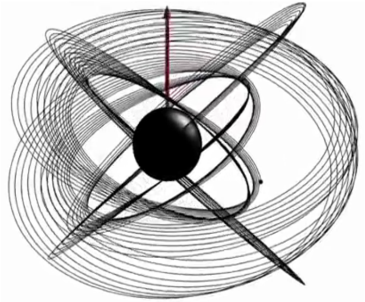

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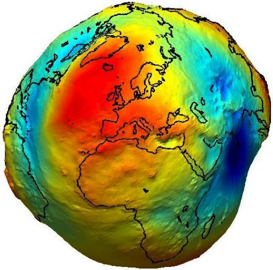
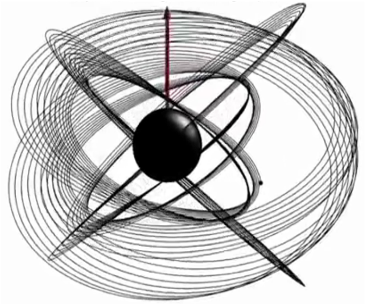
Test of the black hole no-hair theorem

Geodesy



M_ℓ arbitrary

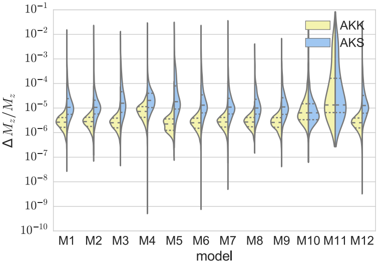
Botriomeladesy

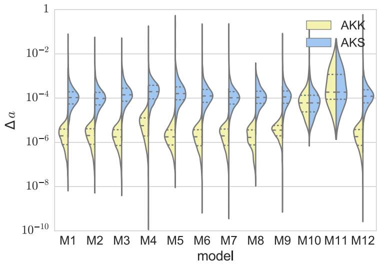


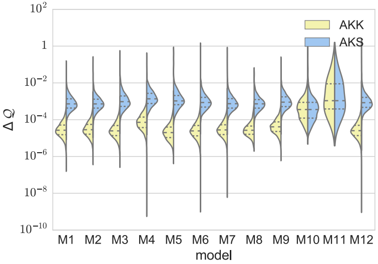
(Credit: S. Drasco)

$$M_\ell + iS_\ell = M(ia)^\ell$$

Testing the black hole no-hair theorem



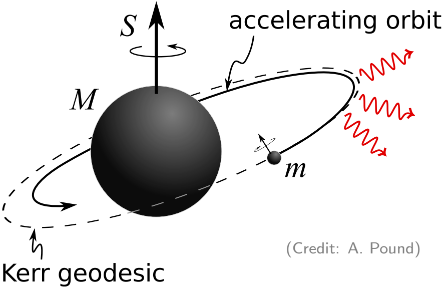




$$Q = -S^2/M$$

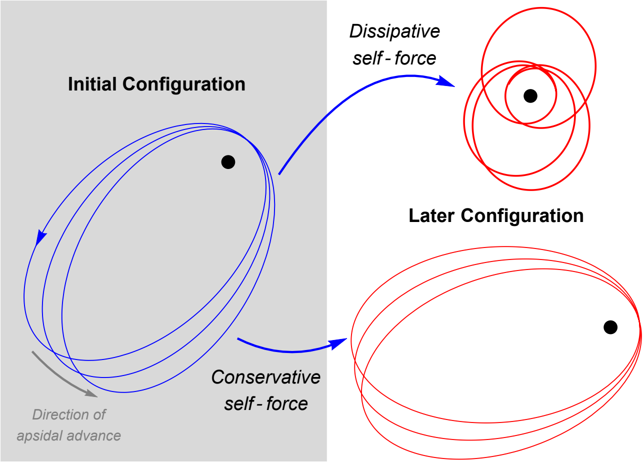
[PRD **95** (2017) 103012]

Gravitational self-force



- Dissipative component $\longleftrightarrow$ gravitational waves
- Conservative component $\longleftrightarrow$ some secular effects

Gravitational self-force

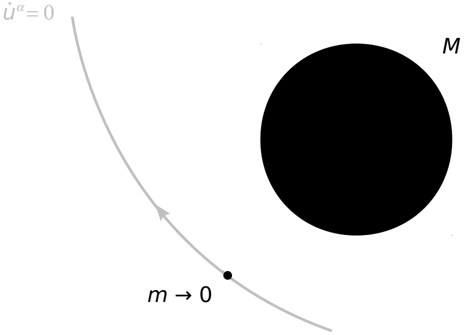


(Credit: Osburn *et al.* 2016)

Gravitational self-force

Spacetime metric

$$\mathrm{g}_{\alpha\beta} = g_{\alpha\beta}$$



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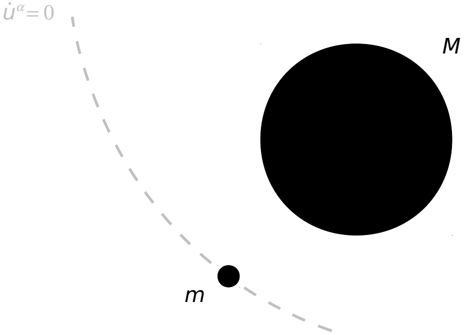
Small parameter

$$q \equiv \frac{m}{M} \ll 1$$



Gravitational self-force

Spacetime metric

$$\mathrm{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$$

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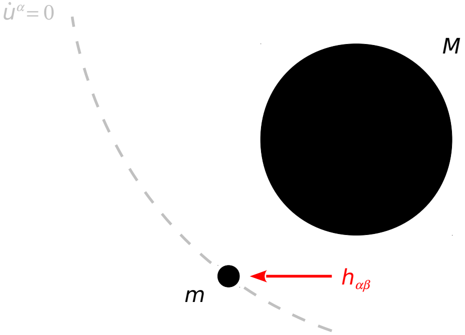

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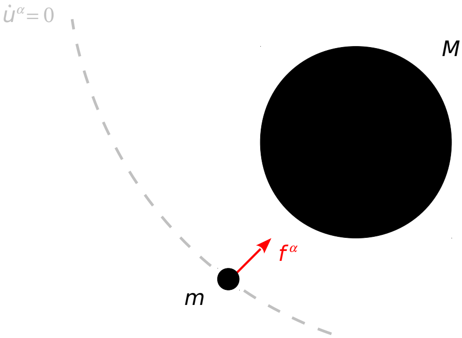

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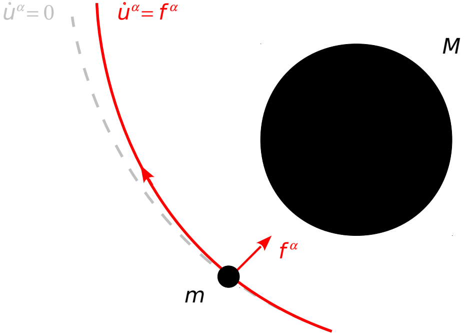
Gravitational self-force

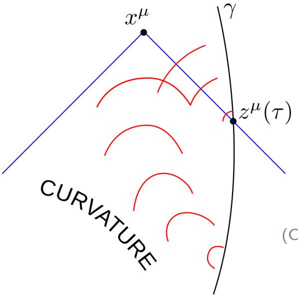
$$\dot{u}^\alpha \equiv u^\beta \nabla_\beta u^\alpha = f^\alpha[h]$$



Equation of motion

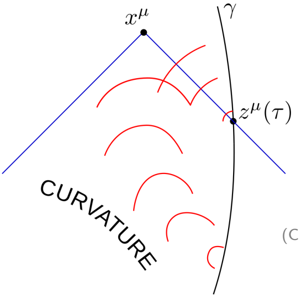
Metric perturbation

$$h_{\alpha\beta} = h_{\alpha\beta}^{\text{direct}} + h_{\alpha\beta}^{\text{tail}}$$



(Credit: A. Pound)

Equation of motion



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Metric perturbation

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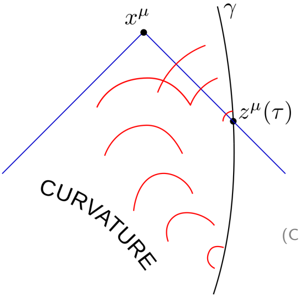
MiSaTaQuWa equation

$$\dot{u}^\alpha = \underbrace{\left(g^{\alpha\beta} + u^\alpha u^\beta\right)}_{\text{projector} \perp u^\alpha} \underbrace{\left(\tfrac{1}{2}\nabla_\beta h^{\text{tail}}_{\lambda\sigma} - \nabla_\lambda h^{\text{tail}}_{\beta\sigma}\right) u^\lambda u^\sigma}_{\text{``force''}}$$

Equation of motion



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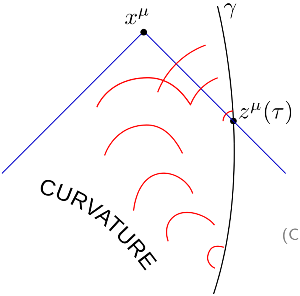

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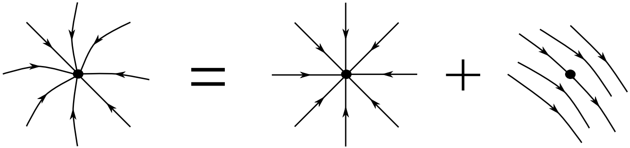
Beware: the self-force is *gauge-dependant*

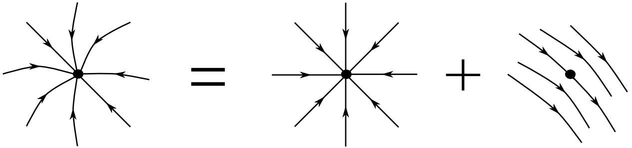
Generalized equivalence principle



(Credit: A. Pound)

body's field $h_{\alpha\beta}$ singular field $h^{S}_{\alpha\beta}$ regular field $h^{R}_{\alpha\beta}$

Generalized equivalence principle



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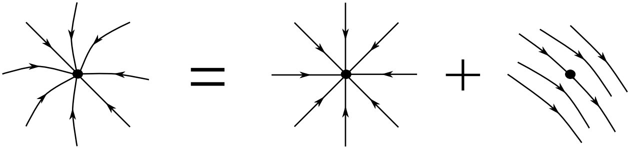
$$h^S \sim m/r$$

$$\Box h^S \sim -16\pi T$$

$$f^\alpha[h^S] = 0$$

Generalized equivalence principle



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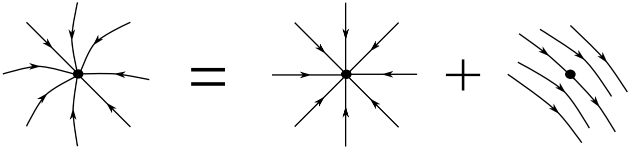
Generalized equivalence principle



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self-acc. motion in $g_{\alpha\beta}$ $\Longleftrightarrow$ **geodesic motion** in $g_{\alpha\beta} + h^R_{\alpha\beta}$

Outline

1. Gravitational waves
2. EMRIs and the gravitational self-force
3. Geodesic motion in Kerr spacetime
4. Beyond the geodesic approximation
5. Innermost stable circular orbits

Hamiltonian formulation

Canonical Hamiltonian

$$H(x, u) = \frac{1}{2}\, g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

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Equations of motion

$$\dot{x}^\alpha = \frac{\partial H}{\partial u_\alpha} \,, \quad \dot{u}_\alpha = -\frac{\partial H}{\partial x^\alpha}$$

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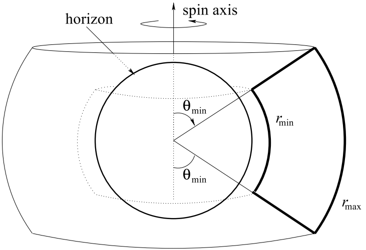

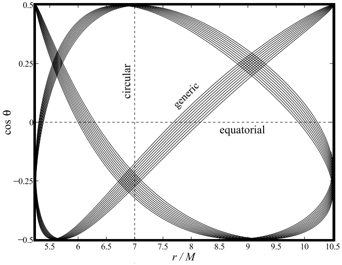

Generalized action-angle variables

[Carter, PRD 1968; Schmidt, CQG 2002]

- The Hamilton-Jacobi equation is *completely separable*

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- Canonical transfo. to *coordinate-invariant* action-angle variables (w^α, J_α), with $w^i + 2\pi \equiv w^i$ and

$$J_t = \frac{1}{2\pi} \int_0^{2\pi} u_t \,\mathrm{d}t = -E \,, \quad J_r = \frac{1}{2\pi} \oint u_r(r) \,\mathrm{d}r$$

$$J_\phi = \frac{1}{2\pi} \oint u_\phi \,\mathrm{d}\phi = L \,, \qquad J_\theta = \frac{1}{2\pi} \oint u_\theta(\theta) \,\mathrm{d}\theta$$

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$$J_\phi = \frac{1}{2\pi}\oint u_\phi \, \mathrm{d}\phi = L \,, \quad J_\theta = \frac{1}{2\pi}\oint u_\theta(\theta)\,\mathrm{d}\theta$$

- Hamilton's canonical equations of motion for $H(J)$:

$$\dot{w}^\alpha = \left(\frac{\partial H}{\partial J_\alpha}\right)_w \equiv \omega^\alpha \,, \quad \dot{J}_\alpha = -\left(\frac{\partial H}{\partial w^\alpha}\right)_J = 0$$

Hamiltonian first law of mechanics

[Le Tiec, CQG 2014]

- Varying $H(J)$ and using Hamilton's equations, as well as the on-shell constraint $H = -\frac{1}{2}$, yields the variational formula

$$\delta H = \omega^\alpha \delta J_\alpha = 0$$

Hamiltonian first law of mechanics

[Le Tiec, CQG 2014]

- Varying $H(J)$ and using Hamilton's equations, as well as the on-shell constraint $H = -\frac{1}{2}$, yields the variational formula

$$\delta H = \omega^\alpha \delta J_\alpha = 0$$

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- Using non-specific variables $\mathcal{J}_\alpha \equiv m J_\alpha$ and the fundamental t-frequencies $\Omega^\alpha \equiv \omega^\alpha / \omega^t \equiv z\, \omega^\alpha$, this gives

$$\delta\mathcal{E} = \Omega^\varphi\, \delta\mathcal{L} + \Omega^r\, \delta\mathcal{J}_r + \Omega^\theta\, \delta\mathcal{J}_\theta + z\, \delta m$$

Outline

Hamiltonian formulation

- We consider only the *conservative piece* of the self-force:

$$h^R_{\alpha\beta} = \frac{1}{2}\left(h^{\text{ret}}_{\alpha\beta} + h^{\text{adv}}_{\alpha\beta}\right) - h^S_{\alpha\beta}$$

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- Canonical transform. to action-angle variables (w^α, J_α)
- Hamilton's canonical equations of motion for $H(w,J;\gamma)$:

$$\dot{w}^\alpha = \left(\frac{\partial H}{\partial J_\alpha}\right)_w \equiv \omega^\alpha\,, \quad \dot{J}_\alpha = -\left(\frac{\partial H}{\partial w^\alpha}\right)_J \neq 0$$

No secular change in the actions

- For any function f of the canonical variables (x, u) we define the long (proper) time average

$$\langle f \rangle \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x(\tau), u(\tau))\,\mathrm{d}\tau$$

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- The *average* rate of change of the actions vanishes:

$$\langle \dot{J}_\alpha \rangle = -\left\langle \left(\frac{\partial H_{(1)}}{\partial w^\alpha} \right)_J \right\rangle = 0$$

Gauge transformations

- A gauge transformation $x^\alpha \to x^\alpha + \xi^\alpha$ induces a canonical transformation with generator $\Xi(x,u) = u_\alpha \xi^\alpha(x)$ such that

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- Despite that, we have the *gauge-invariant* identity:

$$\dot{w}^\alpha J_\alpha = \omega^\alpha J_\alpha = -1$$

Gauge-invariant information

- Neither the action variables nor the fundamental frequencies are gauge invariant:

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- The relationship between the average redshift $z \equiv 1/\left\langle \omega^t \right\rangle$ and the average t-frequencies $(\Omega^r, \Omega^\theta, \Omega^\phi)$ is gauge-invariant

A special gauge choice

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- These gauge conditions can indeed be imposed *consistently*

An effective Hamiltonian

- In this particular gauge, Hamilton's equations of motion take the remarkably simple form

$$\dot{w}^\alpha = \omega^\alpha = \omega^\alpha_{(0)}(J) + \frac{1}{2}\frac{\partial H_{\text{int}}}{\partial J_\alpha}\,, \quad \dot{J}_\alpha = 0$$

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- Despite the freedom to change the constant actions J_α by a gauge transformation, this effective Hamiltonian is *unique*

First law of mechanics

- Using the effective Hamiltonian $\mathcal{H}(J)$, the test-mass *first law of mechanics* can be extended to relative $\mathcal{O}(q)$:

$$\delta\tilde{\mathcal{E}} = \Omega^\varphi\,\delta\tilde{\mathcal{L}} + \Omega^r\,\delta\tilde{\mathcal{J}}_r + \Omega^\theta\,\delta\tilde{\mathcal{J}}_\theta + z\,\delta m$$

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- It involves the *renormalized* actions and the average redshift

$$\tilde{\mathcal{J}}_\alpha \equiv m\tilde{J}_\alpha \equiv mJ_\alpha\Big(1 - \tfrac{1}{2}H_{\text{int}}\Big)$$

$$z = z_{(0)} + z_{(1)} = z_{(0)}\big(1 + H_{\text{int}}\big)$$

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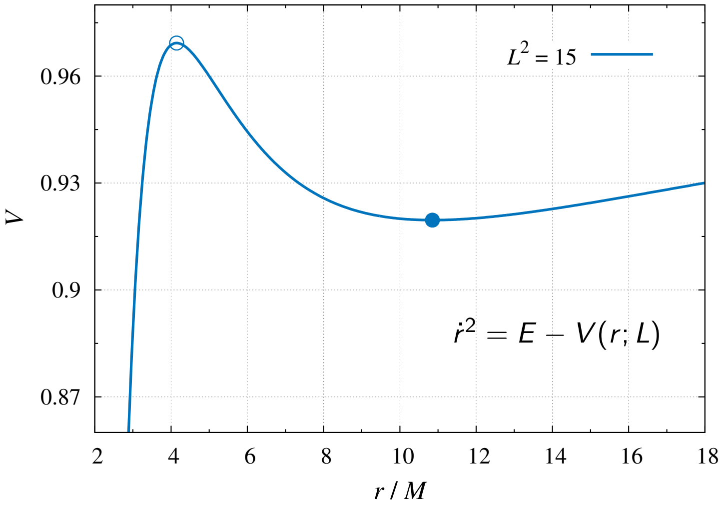
- The actions $\tilde{\mathcal{J}}_\alpha$ and the average redshift z, as functions of $(\Omega^r, \Omega^\theta, \Omega^\phi)$, include conservative self-force corrections from the *gauge-invariant* interaction Hamiltonian H_{int}

Outline

Innermost stable circular orbit (ISCO)



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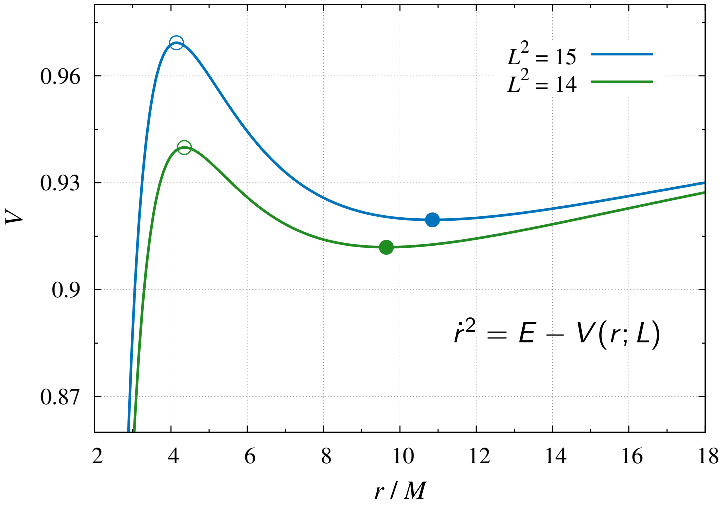

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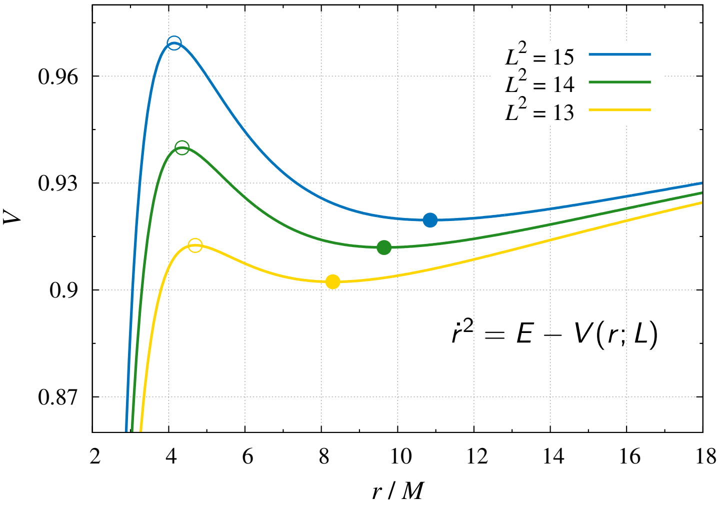

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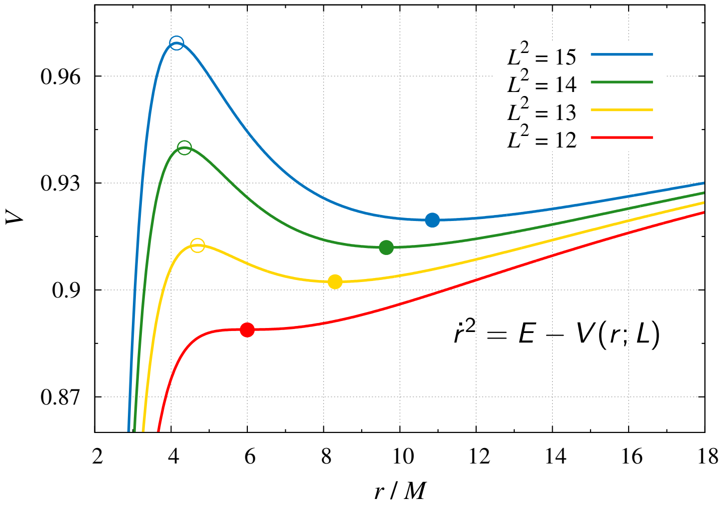


Innermost stable circular orbit (ISCO)

- The innermost stable circular orbit is identified by a vanishing restoring radial force under small-e perturbations:

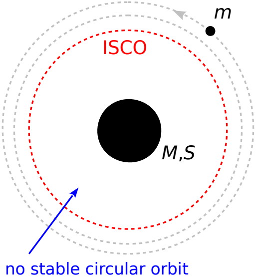
$$\frac{\partial^2 H}{\partial r^2} = 0 \quad \longrightarrow \quad \Omega_{\text{isco}}$$

- The minimum energy circular orbit is the most bound orbit along a sequence of circular orbits:

$$\frac{\partial E}{\partial \Omega} = 0 \quad \longrightarrow \quad \Omega_{\text{meco}}$$

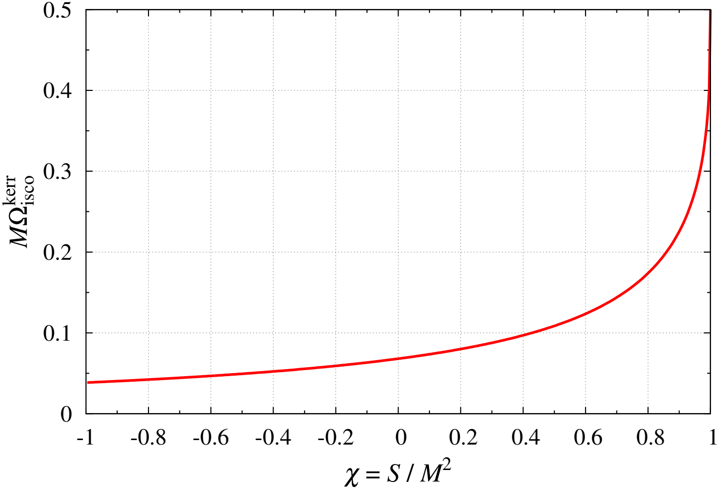
- For Hamiltonian systems, it can be shown that

$$\Omega_{\text{isco}} = \Omega_{\text{meco}}$$



Kerr ISCO frequency *vs* black hole spin

[Bardeen *et al.*, ApJ 1972]



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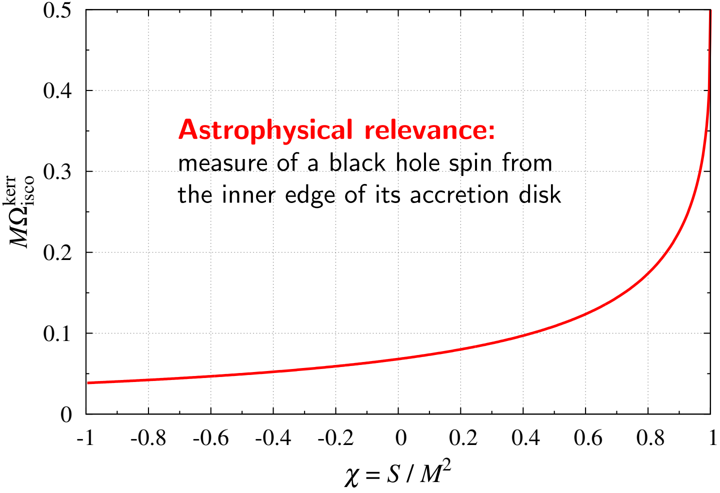

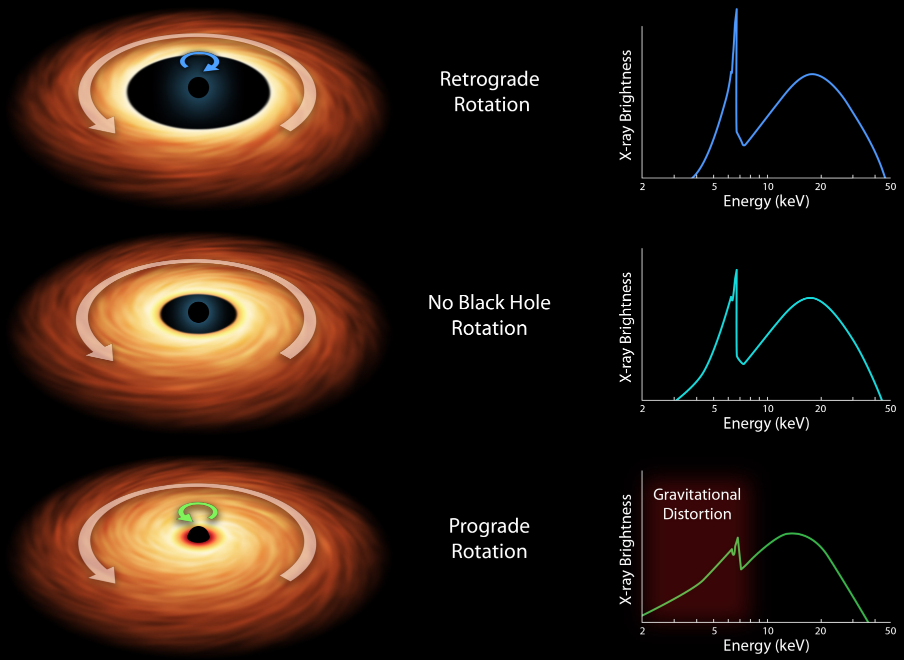
Retrograde Rotation
X-ray Brightness
2
5
10
20
50
Energy (keV)
No Black Hole Rotation
X-ray Brightness
2
5
10
20
50
Energy (keV)
Prograde Rotation
Gravitational Distortion
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5
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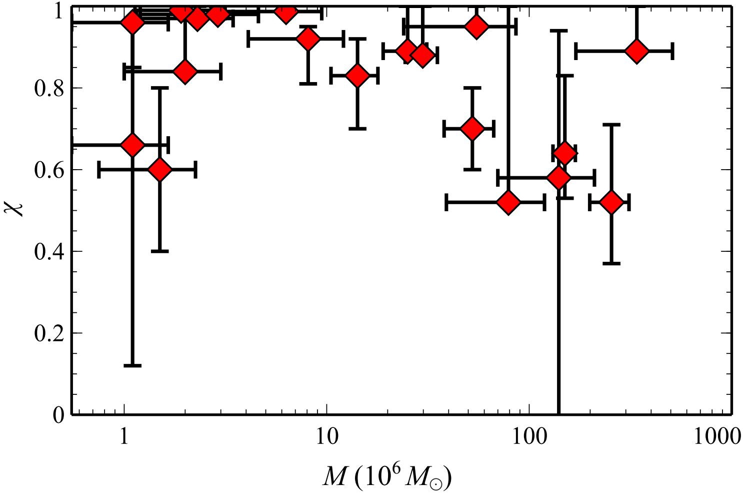
Spins of supermassive black holes

[Reynolds, CQG 2013]



Frequency shift of the Kerr ISCO

[Isoyama *et al.*, PRL 2014]

- The orbital frequency of the Kerr ISCO is shifted under the effect of the conservative self-force:

$$(M+m)\Omega_{\text{isco}} = \underbrace{M\Omega^{(0)}_{\text{isco}}(\chi)}_{\substack{\text{test mass}\\ \text{result}}} \left\{ 1 + \underbrace{q\, C_\Omega(\chi)}_{\substack{\text{self-force}\\ \text{correction}}} + \mathcal{O}(q^2) \right\}$$

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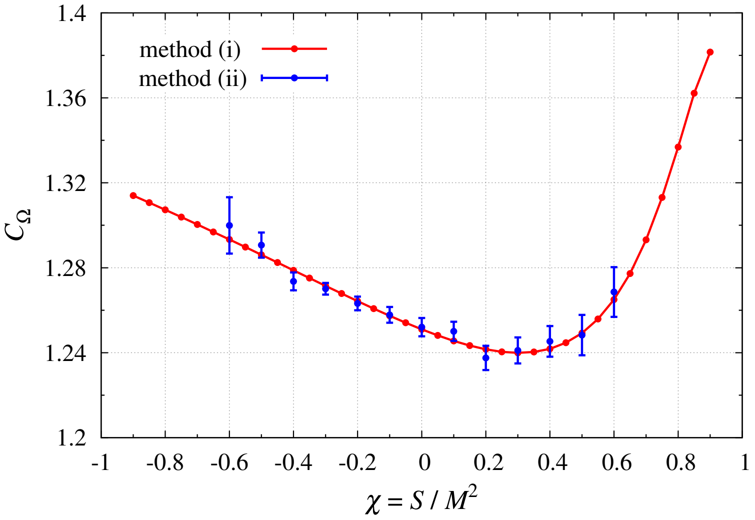
- Combining the Hamiltonian first law with the MECO conditio $\partial E/\partial\Omega = 0$ yields the same result:

$$C_\Omega = \frac{1}{2}\,\frac{z''_{(1)}(\Omega^{(0)}_{\text{isco}})}{E''_{(0)}(\Omega^{(0)}_{\text{isco}})}$$

ISCO frequency shift *vs* black hole spin

[Isoyama *et al.*, PRL 2014]



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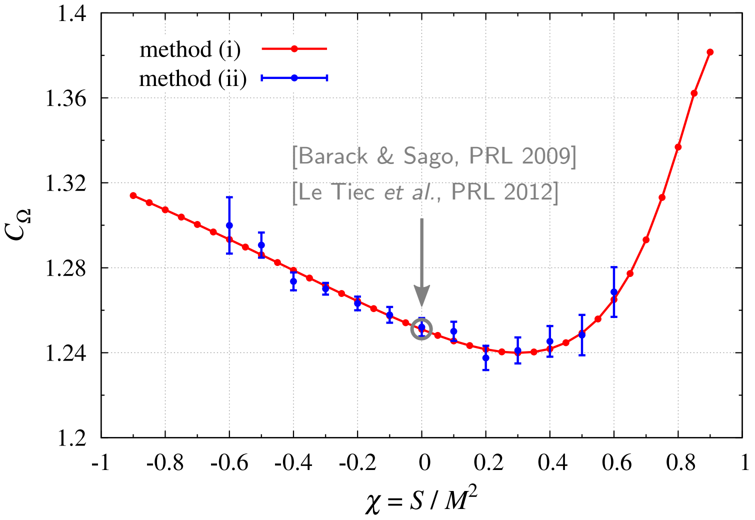

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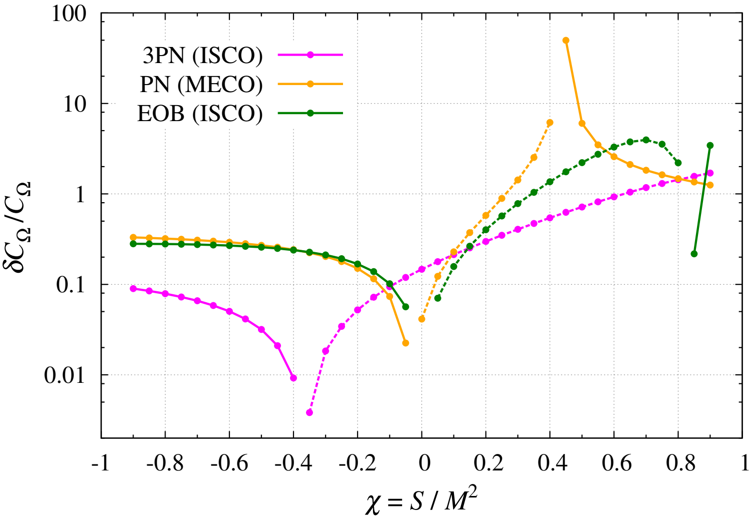

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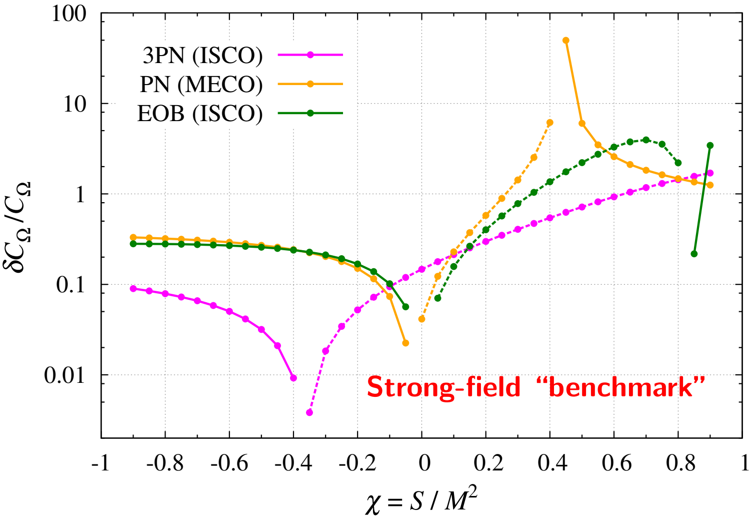

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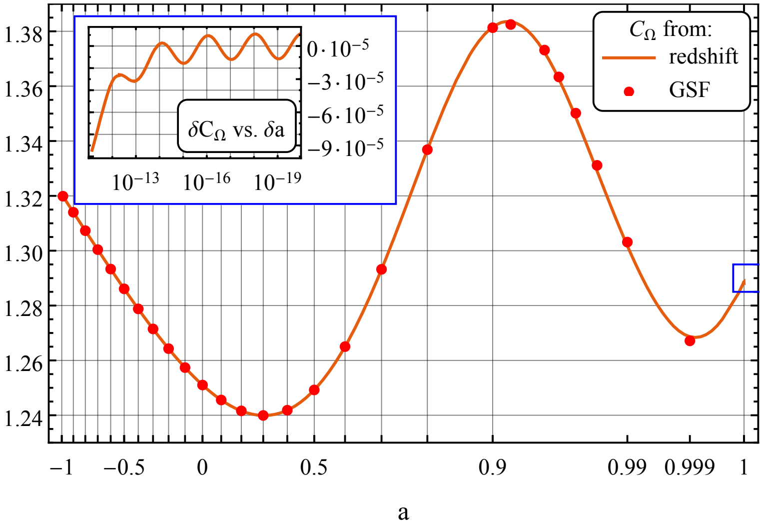
[van de Meent, PRL 2017]



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- We computed the shift in the Kerr ISCO frequency induced by the conservative piece of the SF
- This result provides an accurate strong-field "benchmark" for comparison with other analytical methods (PN, EOB)