Tidal deformations of black holes

Alexandre Le Tiec

Laboratoire Univers et Théories
Observatoire de Paris / CNRS

Collaboration with M. Casals and E. Franzin

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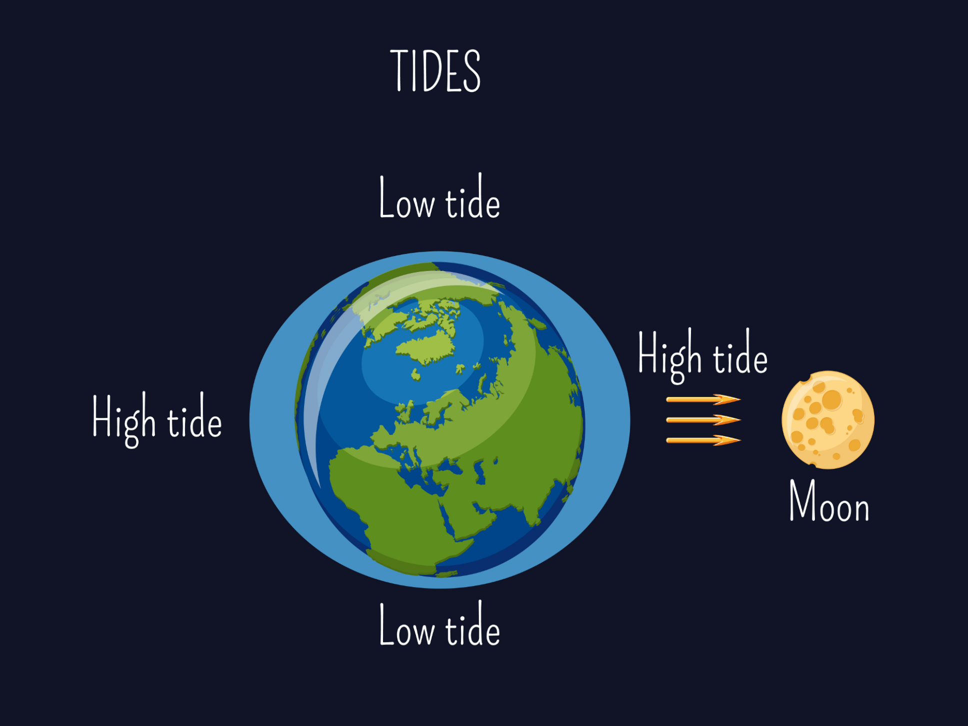
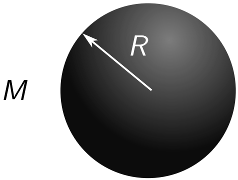
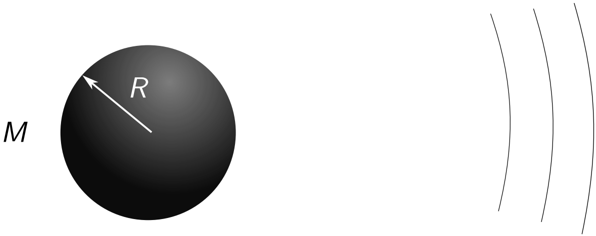

TIDES
Low tide
High tide
High tide
Moon
Low tide

Newtonian theory of Love numbers



$$U = \frac{M}{r}$$

Newtonian theory of Love numbers



$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(\mathbf{0})$$

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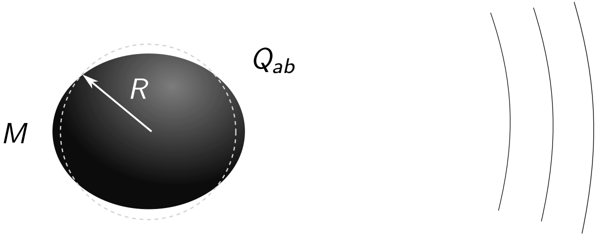

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$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} + \frac{3}{2} \frac{x^a x^b Q_{ab}}{r^5}$$

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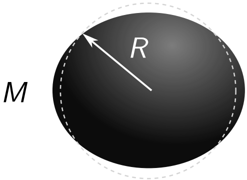

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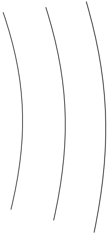

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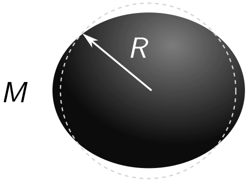
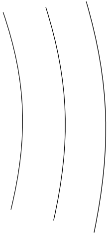


$$Q_{ab} = \lambda_2 \mathcal{E}_{ab} = -\frac{2}{3} k_2 R^5 \mathcal{E}_{ab}$$



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Newtonian theory of Love numbers

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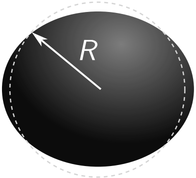

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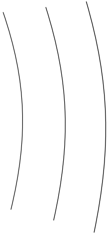

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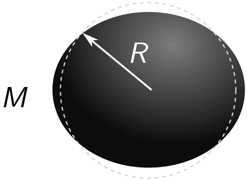
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} \left[1 + 2 k_2 \left(\frac{R}{r} \right)^5 \right]$$

Newtonian theory of Love numbers



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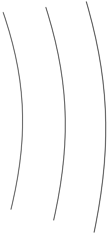

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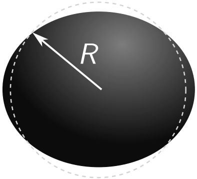

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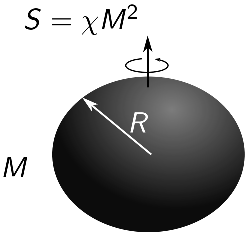
Newtonian theory of Love numbers



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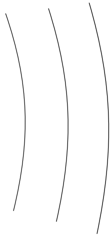

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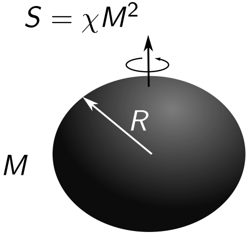
$$k_{\ell m} = k_\ell^{(0)} + i m \chi \, k_\ell^{(1)} + O(\chi^2)$$

Newtonian theory of Love numbers



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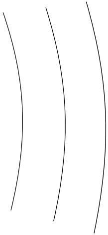

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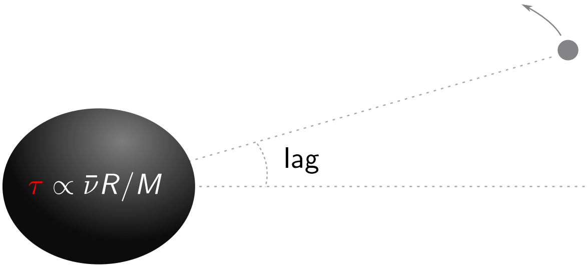
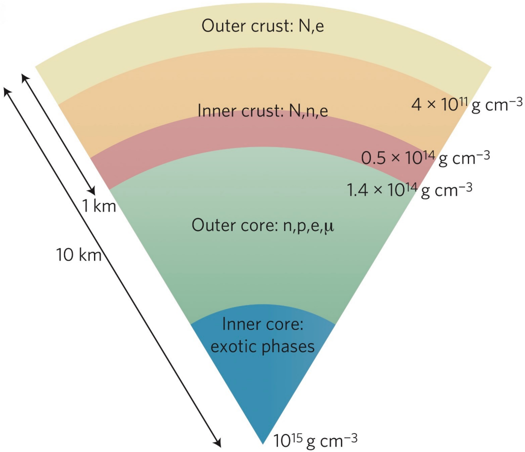
Tidal Love numbers $k_{\ell m} \longleftrightarrow$ **body's internal structure**

Tidal dissipation: lag, heating and torquing



$$Q_{ab}(t) = -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t) - \tau \dot{\mathcal{E}}_{ab}(t) + \cdots \right]$$
$$= -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t - \tau) + \cdots \right]$$

Internal structure of neutron stars

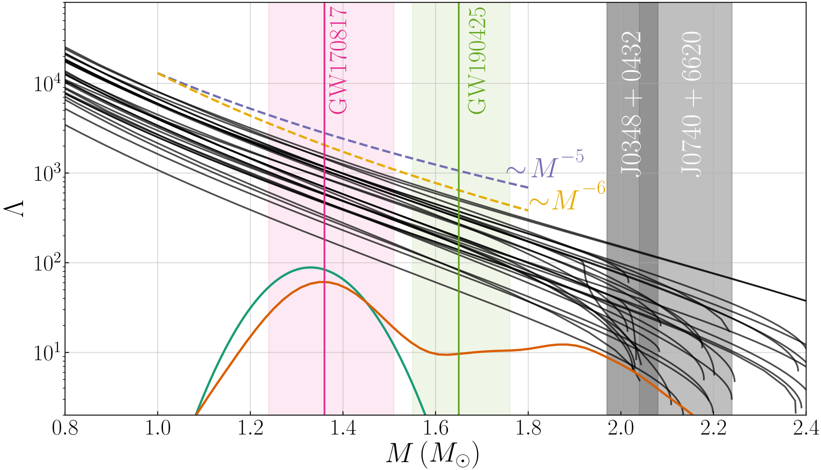


GW observations as probes of **neutron star internal structure**

Gravitational-wave observations

[Chatziioannou, GRG 2020]

$\Lambda \equiv \lambda_2/M^5 \propto k_2/C^5$



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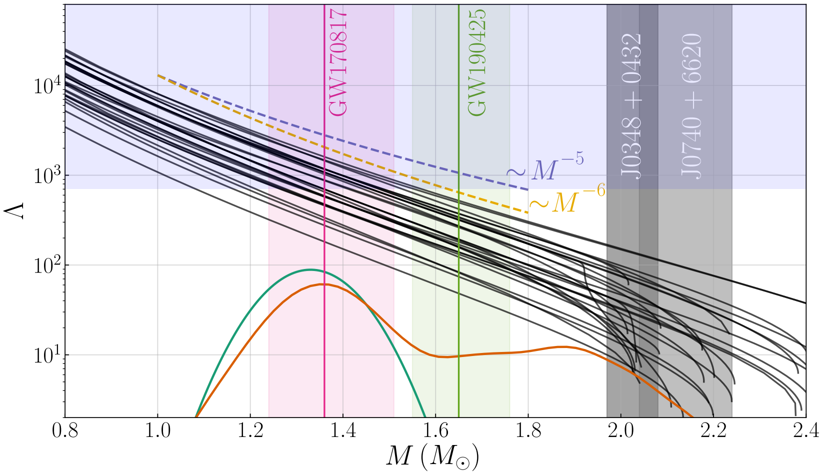
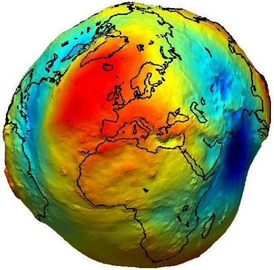
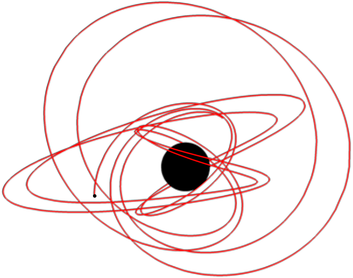


Do isolated black holes have hair?

Geodesy

Botromeladesy



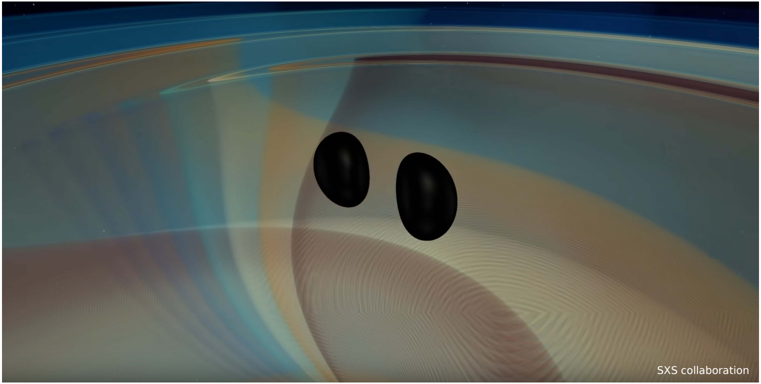


$M_{\ell m}$ arbitrary

$M_{\ell 0} + iS_{\ell 0} = M(ia)^{\ell}$

Do tidally-interacting black holes deform?

Black hole **tomography** by gravitational-wave observations



Relativistic theory of Love numbers

- Electric-type and magnetic-type tidal moments:

$$\mathcal{E}_{a_1\cdots a_\ell} \propto \left[C_{0a_10a_2;a_3\cdots a_\ell}\right]^{\mathsf{STF}}, \quad \mathcal{B}_{a_1\cdots a_\ell} \propto \left[\varepsilon_{a_1bc}C_{a_20bc;a_3\cdots a_\ell}\right]^{\mathsf{STF}}$$

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- Metric and Geroch-Hansen multipole moments:

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + \underbrace{h^{\mathsf{tidal}}_{\alpha\beta}}_{\sim r^\ell} + \underbrace{\color{red}{h^{\mathsf{resp}}_{\alpha\beta}}}_{\sim r^{-(\ell+1)}} \quad \longrightarrow \quad \begin{cases} M_{\ell m} = \mathring{M}_{\ell m} + \color{red}{\delta M_{\ell m}} \\ S_{\ell m} = \mathring{S}_{\ell m} + \color{red}{\delta S_{\ell m}} \end{cases}$$

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- *Four* families of tidal deformability parameters:

$$\lambda^{M\mathcal{E}}_{\ell\ell' m} \equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{E}_{\ell' m}} \qquad \lambda^{S\mathcal{B}}_{\ell\ell' m} \equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{B}_{\ell' m}}$$

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Love numbers of a spinning black hole

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- The associated dimensionless tidal Love numbers are

$$k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} \doteq -\frac{im\chi}{120} \quad \text{and} \quad k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} \doteq 0$$

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- For a dimensionless black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \simeq 2 \times 10^{-3} \quad \longrightarrow \quad \text{black holes are } \textbf{“rigid”}$$

Love tensor of a spinning black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\mathrm{el}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\mathrm{mag}} \mathcal{B}_{ab}$$

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- For a spinning black hole we have the more general tensorial relations

$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}_{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}_{cd}$$

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$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}_{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}_{cd}$$

- To linear order in the black hole spin vector S^a we find

$$\delta M_{ab} = \frac{16}{45}\, M^3\, S^c \mathcal{E}^d{}_{(a} \varepsilon_{b)cd}$$
$$\delta S_{ab} = \frac{16}{45}\, M^3\, S^c \mathcal{B}^d{}_{(a} \varepsilon_{b)cd}$$

Summary

- Love numbers of spinning black holes do not vanish in general
- We computed in closed-form the leading (quadrupolar) Love numbers to linear order in the black hole spin
- Kerr black holes deform like any other self-gravitating body, despite being particularly “rigid” compact objects
- This is closely related to the phenomenon of tidal torquing
- New black hole test of the Kerr-like nature of the massive compact objects at the center of galaxies?

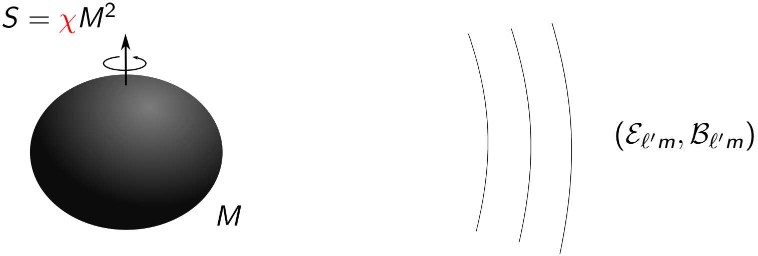
Spinning black holes fall in Love!

Additional Material

Recent work on black hole Love numbers

Reference	Background	Tidal field
[Binnington & Poisson 2009]	Schwarzschild	weak, generic ℓ
[Damour & Nagar 2009]	Schwarzschild	weak, generic ℓ
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $\ell = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(S^2)$	weak, $(\ell, m) = (2, 0)$
[Le Tiec & Casals 2021]	Exact Kerr	weak, generic ℓ
[Chia 2021]	Exact Kerr	weak, generic ℓ
[Goldberger et al. 2021]	Exact Kerr	weak, generic ℓ
[Charalambous et al. 2021]	Exact Kerr	weak, generic ℓ
[Hui 2021]	Schwar-(A)dS	weak, generic ℓ

Investigating Kerr's Love



- Metric reconstruction through the Hertz potential Ψ:

$$(\mathcal{E}_{\ell' m}, \mathcal{B}_{\ell' m}) \to \psi_0 \to \Psi \to h_{\alpha\beta} \to (M_{\ell m}, S_{\ell m}) \to \lambda^{M/S, \mathcal{E}/\mathcal{B}}_{\ell \ell' m}$$

- Quadrupolar tidal Love numbers of a Kerr black hole:

$$\lambda^{M\mathcal{E}}_{2\ell' m} = \lambda^{S\mathcal{B}}_{2\ell' m} \doteq \frac{im\chi}{180} (2M)^5 \, \delta_{\ell' 2} \, , \quad \lambda^{M\mathcal{B}}_{2\ell' m} = \lambda^{S\mathcal{E}}_{2\ell' m} = 0$$

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c\to\infty} \psi_0^{\ell m} \propto \mathcal{E}_{\ell m}\, r^{\ell-2} \left[1 + 2k_{\ell m} \left(R/r\right)^{2\ell+1}\right] {}_2Y_{\ell m}(\theta,\phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{\ell m} \propto \left[\mathcal{E}_{\ell m} + \tfrac{3i}{\ell+1}\, \mathcal{B}_{\ell m}\right] R_{\ell m}(r)\; {}_2Y_{\ell m}(\theta,\phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{\ell m}(r) = \underbrace{r^{\ell-2}\left(1+\cdots\right)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + \kappa_{\ell m}\, \underbrace{r^{-\ell-3}\left(1+\cdots\right)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Kerr black hole linear response

$$R_{\ell m}(r) = \underbrace{R^{\text{tidal}}_{\ell m}(r)}_{\sim r^{\ell-2}} + 2k_{\ell m}\underbrace{R^{\text{resp}}_{\ell m}(r)}_{\sim r^{-(\ell+3)}}$$

- The coefficients $k_{\ell m}$ can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$$k_{\ell m} = -im\chi \frac{(\ell+2)!(\ell-2)!}{4(2\ell+1)!(2\ell)!} \prod_{n=1}^{\ell} \left[n^2(1-\chi^2) + m^2\chi^2\right]$$

- The linear response vanishes identically when:
  - the black hole spin vanishes ($\chi = 0$)
  - the tidal field is axisymmetric ($m = 0$)
- Reconstruct the Kerr black hole response $h^{\text{resp}}_{\alpha\beta}$ via Ψ^{resp}

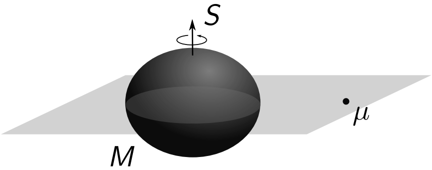
Love tensor of a Kerr black hole

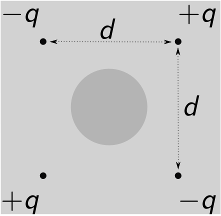
$$(\lambda_{abcd}) \doteq \frac{\chi}{180}\,(2M)^5 \begin{pmatrix} \mathsf{I}_{11} & \mathsf{I}_{12} & \mathsf{I}_{13} \\ \mathsf{I}_{12} & -\mathsf{I}_{11} & \mathsf{I}_{23} \\ \mathsf{I}_{13} & \mathsf{I}_{23} & 0 \end{pmatrix}$$

$$\mathsf{I}_{11} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathsf{I}_{12} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathsf{I}_{13} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \mathsf{I}_{23} \equiv \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

Newtonian static quadrupolar tide





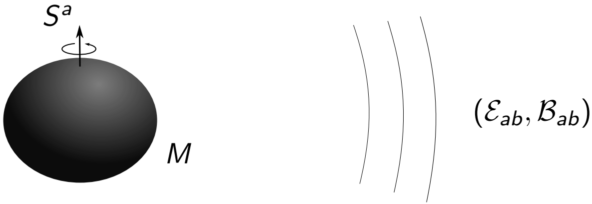
$$\mathcal{E}_{ab} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\delta M_{ab} \doteq 3Q \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\uparrow$$

$$\tfrac{\chi}{180}\,(2M)^5\,\tfrac{\mu}{r^3} = qd^2$$

Tidal torquing of a spinning black hole



- Any spinning body interacting with a tidal environment suffers a tidal torquing [Thorne & Hartle 1980]

$$\langle \dot{S}^a \rangle = -\varepsilon^{abc} \langle M_{bd} \mathcal{E}^d{}_c + S_{bd} \mathcal{B}^d{}_c \rangle$$

- Applied to a spinning black hole this yields

$$\langle \dot{S} \rangle \doteq -\frac{8}{45} M^5 \chi \left[2 \langle \mathcal{E}^{ab} \mathcal{E}_{ab} \rangle - 3 \langle \mathcal{E}_{ab} s^b \mathcal{E}^{ac} s_c \rangle + (\mathcal{E} \to \mathcal{B}) \right]$$

- Full agreement with independant calculation by [Poisson 2004]