Topics in post-Newtonian theory

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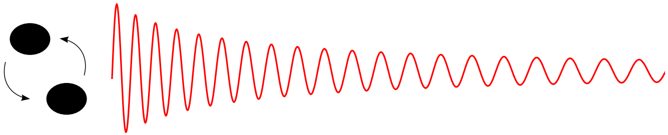

(aLIGO, aVirgo, KAGRA, LISA, ET, SAGE, ...)

Outline

1. Gravitational wave source modelling

2. Post-Newtonian approximation

3. Effective one-body model

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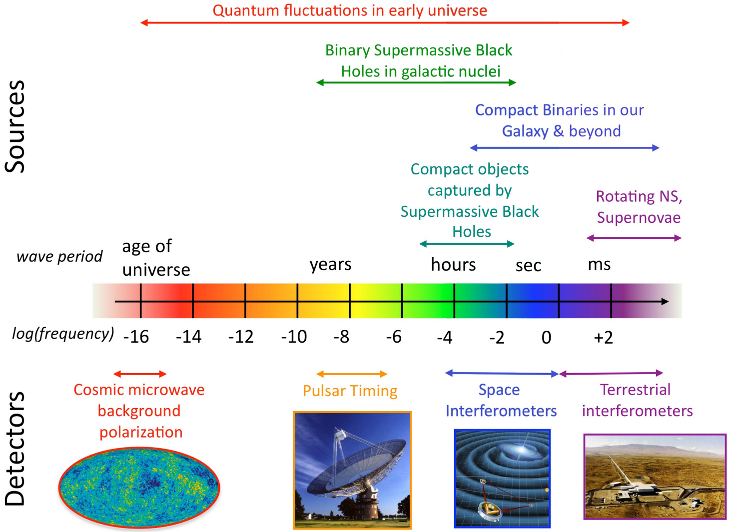
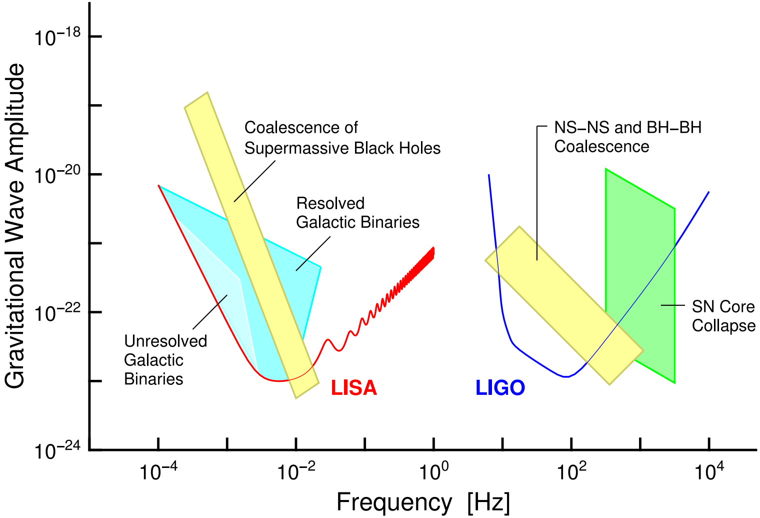
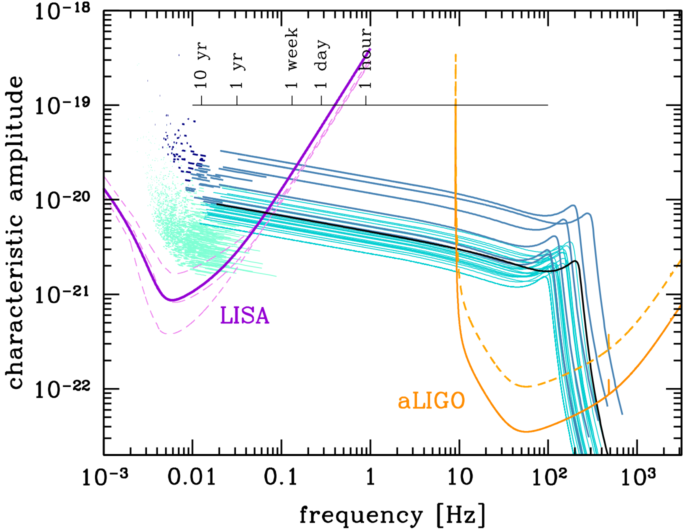
The gravitational-wave spectrum



Promising sources of gravitational waves



Multi-band gravitational-wave astronomy



[PRL **116** (2016) 231102]

Gravitational-wave science

Fundamental physics

- Strong-field tests of GR
- Black hole no-hair theorem
- Cosmic censorship conjecture
- Dark energy equation of state
- Alternatives to general relativity

Astrophysics

- Formation and evolution of compact binaries
- Origin and mechanisms of γ-ray bursts
- Internal structure of neutron stars

Cosmology

- Cosmography and measure of Hubble's constant
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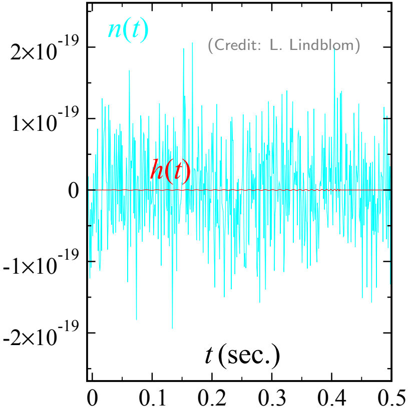
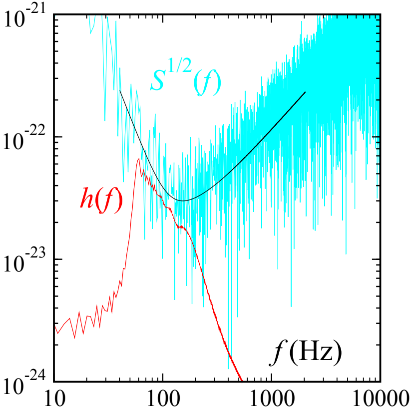
Need for accurate template waveforms





If the expected signal is *known in advance* then $n(t)$ can be filtered and $h(t)$ recovered by matched filtering $\longrightarrow$ **template waveforms**

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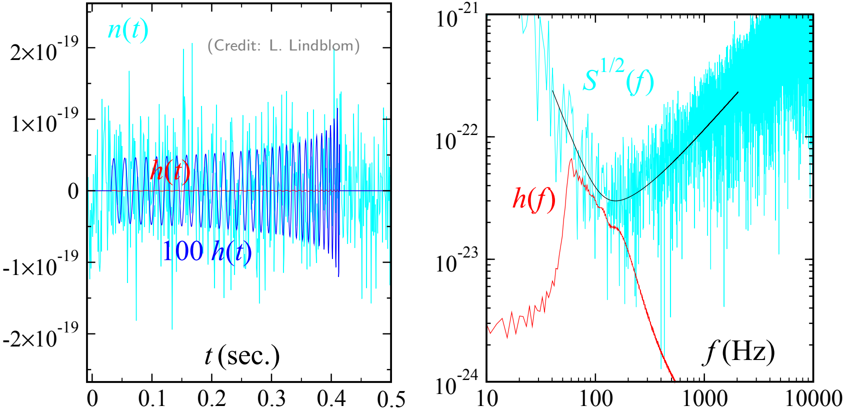

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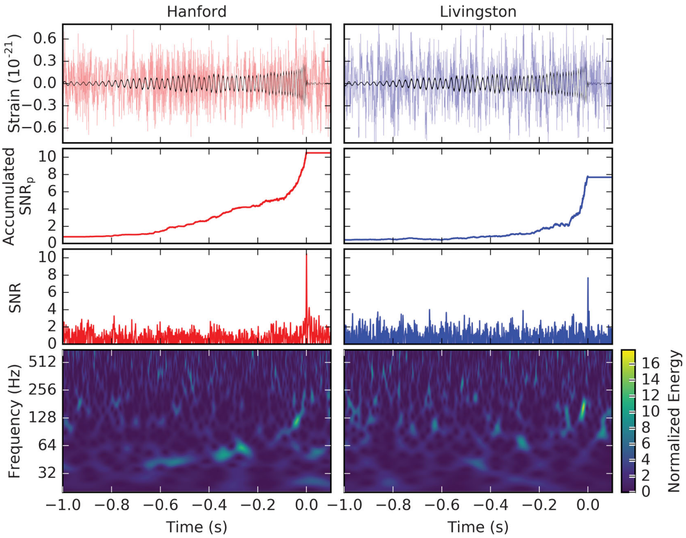
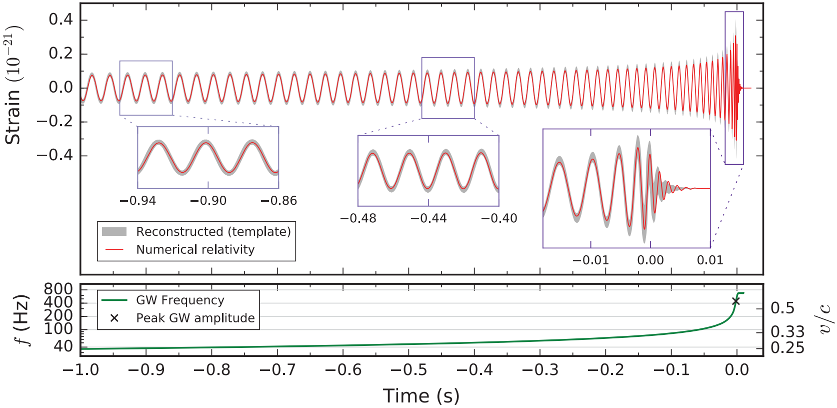
A recent example: the event GW151226



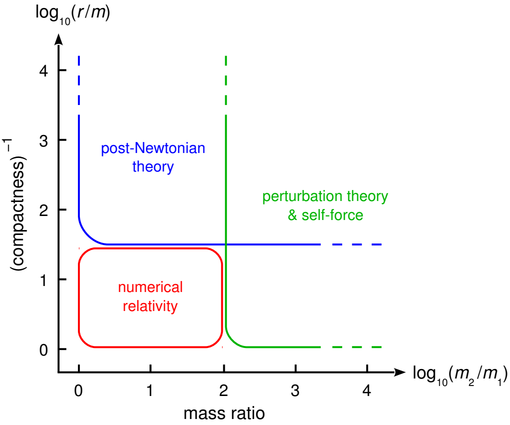
[PRL **116** (2016) 241103]

A long inspiral to merger to ringdown



[PRL **116** (2016) 241103]

Modelling coalescing compact binaries



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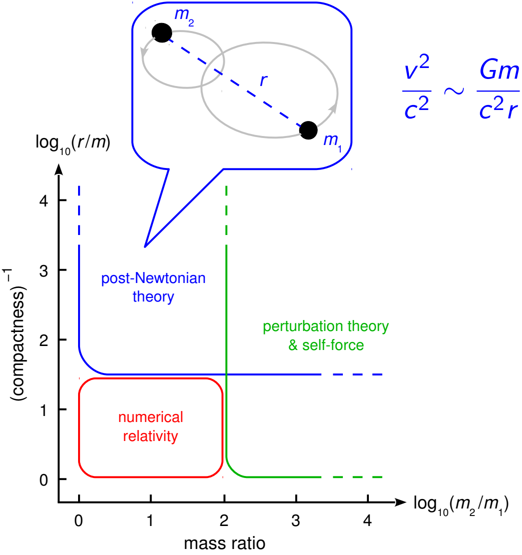
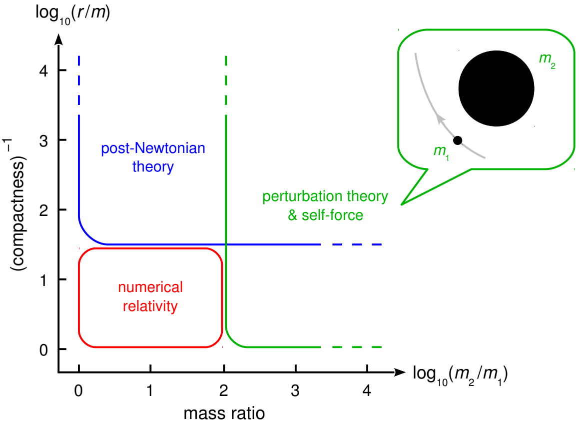


$$\frac{v^2}{c^2} \sim \frac{Gm}{c^2 r} \ll 1$$

Modelling coalescing compact binaries

$$q \equiv \frac{m_1}{m_2} \ll 1$$



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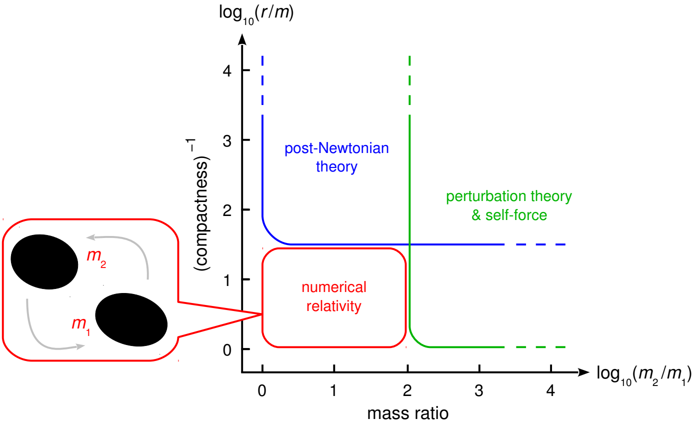

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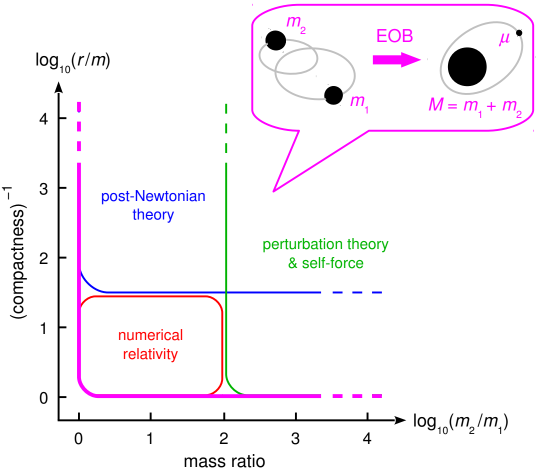
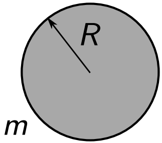


Effacement of the internal structure



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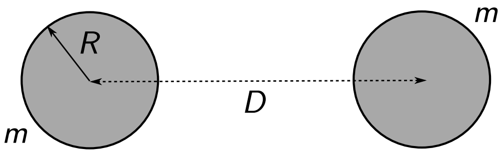

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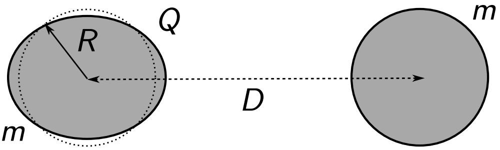

- Induced quadrupole moment: $Q_{ij} \sim R^5\, \partial_i \partial_j U \sim R^5 (Gm/D^3)$

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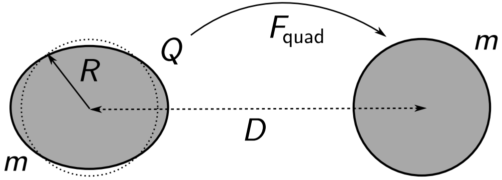

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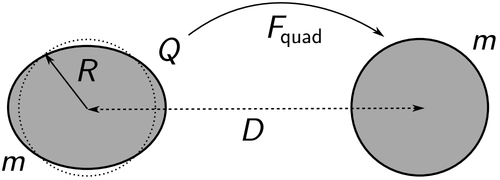

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- For a compact body with $R \sim Gm/c^2$,

$$\frac{F_{\rm quad}}{F_{\rm mono}} \sim \frac{(G^6/c^{10})(m/D)^7}{Gm^2/D^2} \sim \left(\frac{Gm}{c^2 D}\right)^5 \sim \left(\frac{v}{c}\right)^{10} \ll 1$$

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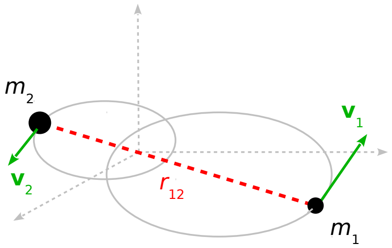
Small parameter

$$\varepsilon \sim \frac{\boldsymbol{v}_{12}^2}{c^2} \sim \frac{Gm}{r_{12}c^2} \ll 1$$



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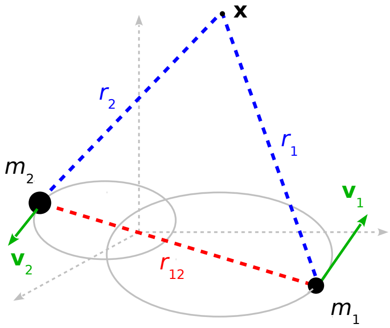

Example

$$g_{00}(t,\boldsymbol{x}) = -1 + \underbrace{\frac{2Gm_1}{r_1 c^2}}_{\text{Newtonian}} + \underbrace{\frac{4Gm_2\boldsymbol{v}_2^2}{r_2 c^4}}_{\text{1PN term}} + \cdots + (1 \leftrightarrow 2)$$

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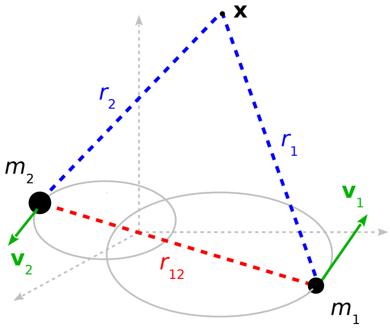

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Notation

nPN order refers to effects $\mathcal{O}(c^{-2n})$ with respect to "Newtonian" solution

Metric potential

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Einstein field equations

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Weak-field approximation

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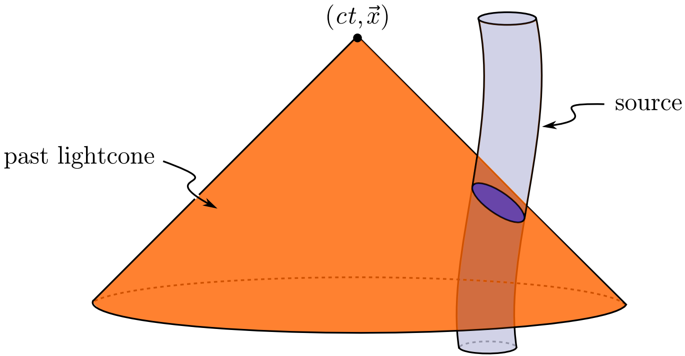
Weak-field approximation

$$|h^{\alpha\beta}| \ll 1 \quad \Longrightarrow \quad \textbf{perturbative nonlinear treatment}$$

Flat space retarded propagator

$$h^{\alpha\beta}(t,\vec{x}) = -4\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\,\tau^{\alpha\beta}(t-|\vec{x}-\vec{y}|/c,\vec{y})$$



Post-Newtonian expansion

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$$\frac{d}{\lambda_{\mathrm{GW}}} \sim \frac{vT}{c(T/2)} \sim \frac{v}{c} \ll 1$$

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$$\frac{S(t-r/c)}{r} = \frac{S(t)}{r} - \frac{1}{c}\,\dot{S}(t) + \frac{r}{2c^2}\,\ddot{S}(t) - \frac{r^2}{6c^3}\,\dddot{S}(t) + \cdots$$

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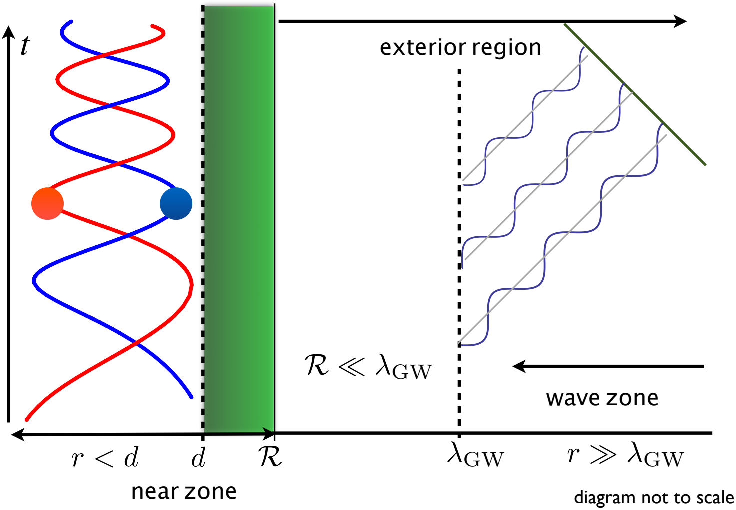
Expansion ill-behaved when $r \gtrsim \lambda_{\mathrm{GW}}$

A wave generation formalism

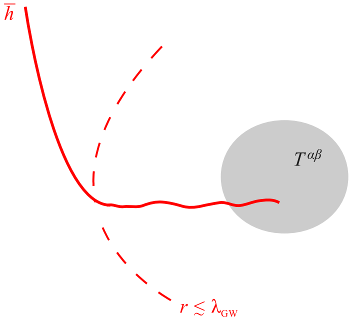


(Credit: Buonanno & Sathyaprakash 2015)

A wave generation formalism

- Post-Newtonian expansion in *near-zone* region $r \ll \lambda_{\mathrm{GW}}$:

$$\overline{h} = \sum_{n \geqslant 0} c^{-n} h^{\mathrm{PN}}_{(n)}$$



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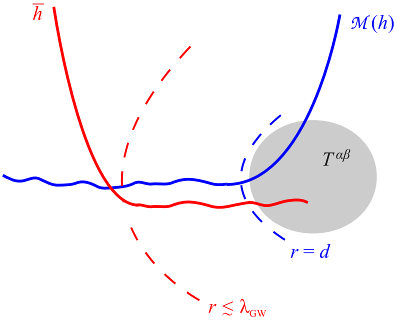

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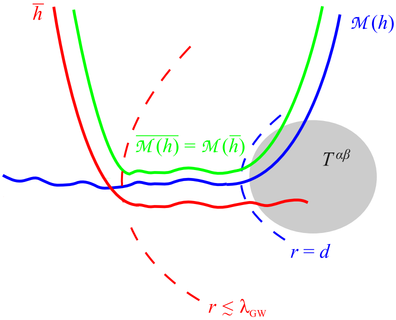

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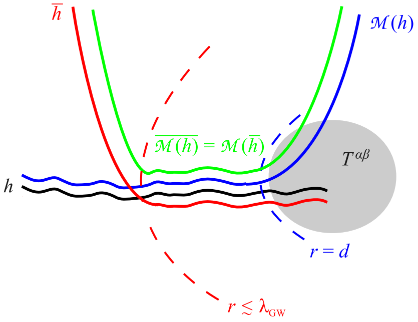

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Extended bodies and point particles

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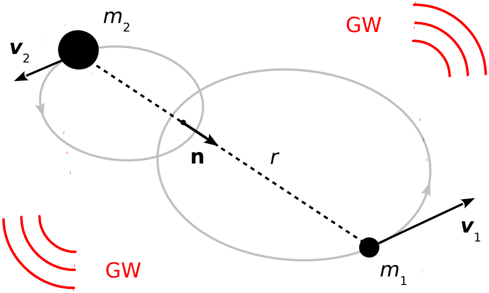
- Divergent self-field of a point particle $\rightarrow$ *regularization* scheme
- Each point mass moves along a geodesic of a *regularized* metric

Two-body equations of motion



$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = \underbrace{-\frac{G m_2}{r^2}\mathbf{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4}}_{\text{conservative terms}} + \underbrace{\frac{\boldsymbol{A}_{2.5\mathrm{PN}}}{c^5}}_{\text{rad. reac.}} + \underbrace{\frac{\boldsymbol{A}_{3\mathrm{PN}}}{c^6}}_{\text{cons. term}} + \underbrace{\frac{\boldsymbol{A}_{3.5\mathrm{PN}}}{c^7}}_{\text{rad. reac.}} + \cdots$$

State of the art: 4PN equations of motion

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = \underbrace{-\frac{Gm_2}{r^2}\mathbf{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4}}_{\text{conservative terms}} + \underbrace{\frac{\boldsymbol{A}_{2.5\mathrm{PN}}}{c^5}}_{\text{rad. reac.}} + \underbrace{\frac{\boldsymbol{A}_{3\mathrm{PN}}}{c^6}}_{\text{cons. term}} + \underbrace{\frac{\boldsymbol{A}_{3.5\mathrm{PN}}}{c^7}}_{\text{rad. reac.}} + \underbrace{\frac{\boldsymbol{A}_{4\mathrm{PN}}}{c^8}}_{\substack{\text{cons. term} \\ +\ \text{rad. tail}}} + \cdots$$

3PN

[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001]	ADM Hamiltonian
[Blanchet & Faye 2001; de Andrade, Blanchet & Faye 2001]	Harmonic EOM
[Itoh & Futamase 2003; Itoh 2004]	Surface integral
[Foffa & Sturani 2011]	Effective field theory

4PN

[Jaranowski & Schäfer 2012, 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
[Bernard, Blanchet, Bohé, Faye, Marchant & Marsat 2015, 2016, 2017]	Fokker Lagrangian
[Foffa, Mastrolia, Sturani & Sturm 2012, 2013, 2017] (partial results)	Effective field theory

Gravitational-wave tail effect at 4PN order

[Blanchet & Damour 1988, Foffa & Sturani 2013, Galley *et al.* 2016]

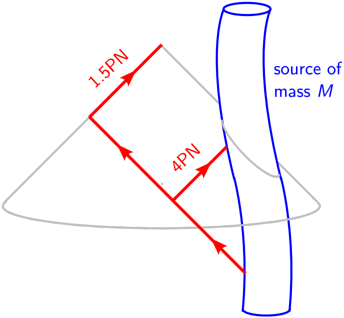
- Starting at 4PN order, the near-zone metric depends on the entire past history of the source:

$$g_{00}^{\text{tail}}(t,\mathbf{x}) = -\frac{8G^2M}{5c^{10}}\,x^ix^j\int_{-\infty}^{t}\mathrm{d}t'\,Q_{ij}^{(7)}(t')\,\ln\left(\frac{c(t-t')}{2r}\right)$$

- This leads to a 4PN non-local-in-time contribution to the Fokker action:

$$S_{\text{F}}^{\text{tail}} = \frac{G^2M}{5c^8}\iint\frac{\mathrm{d}t\,\mathrm{d}t'}{|t-t'|}Q_{ij}^{(3)}(t)Q_{ij}^{(3)}(t')$$

- And to a 1.5PN relative correction to the leading radiation-reaction force



Phasing for inspiralling compact binaries

- Conservative orbital dynamics $\rightarrow$ **4PN** binding energy

$$E(\omega) = \underbrace{-\frac{\mu}{2}\,(m\omega)^{2/3}}_{\substack{\text{Newtonian}\\ \text{binding energy}}} \underbrace{\Big(1 + \cdots\Big)}_{\substack{\text{4PN relative}\\ \text{correction}}}$$

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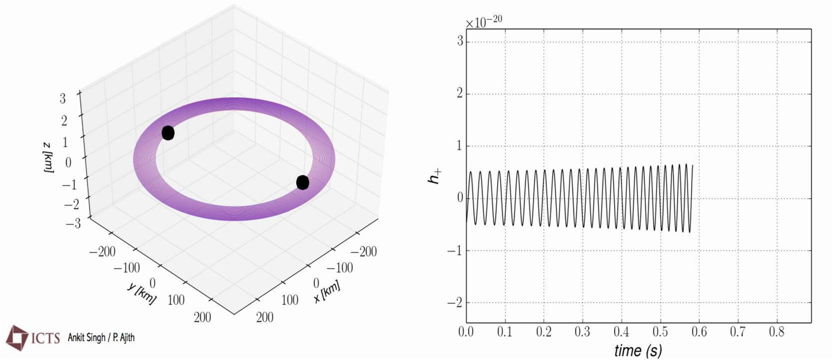
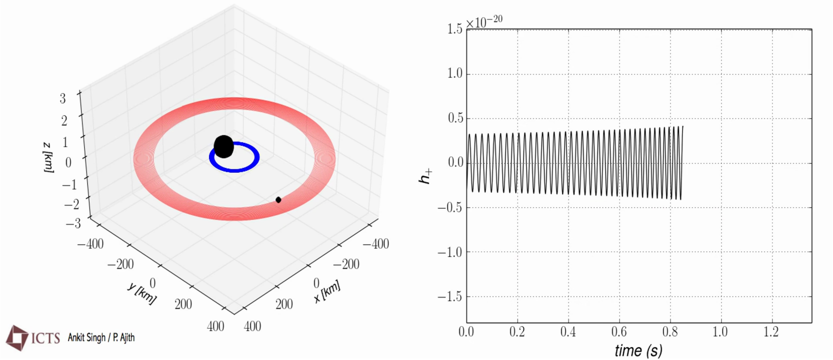
Waveform for inspiralling compact binaries

Equal masses

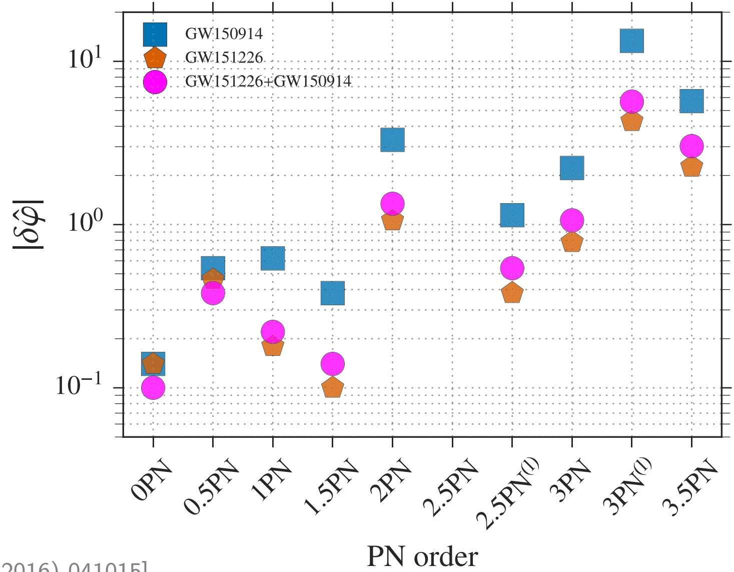


Waveform for inspiralling compact binaries

Unequal masses



Measurement of PN parameters



[PRX **6** (2016) 041015]

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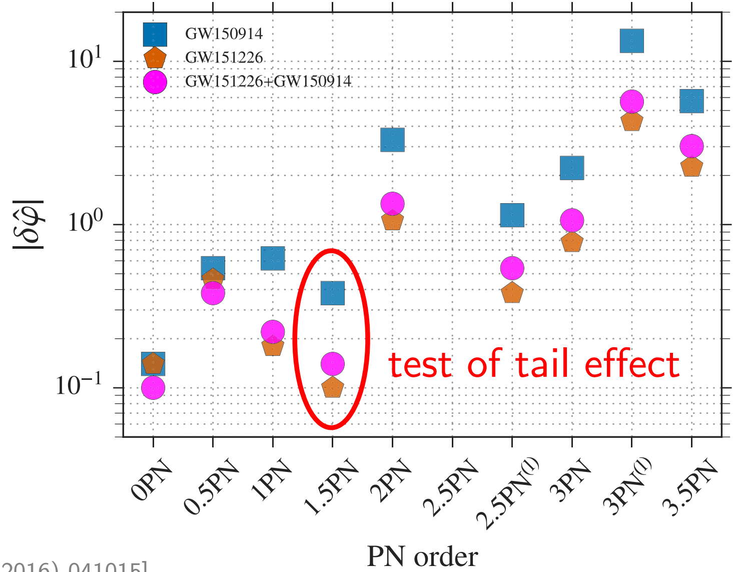
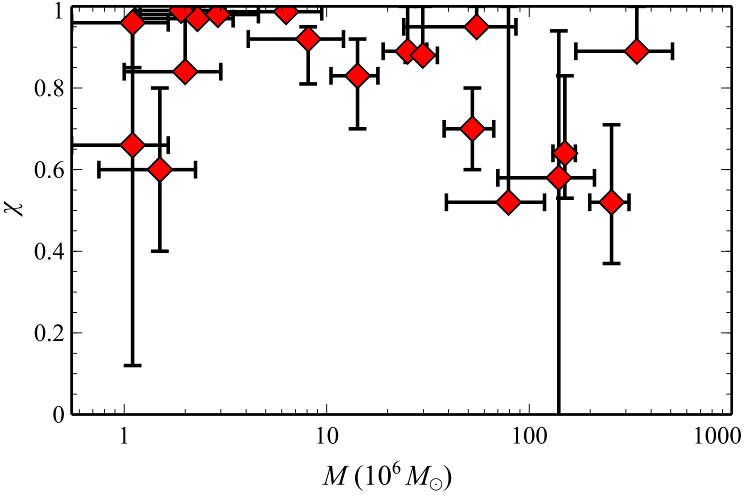


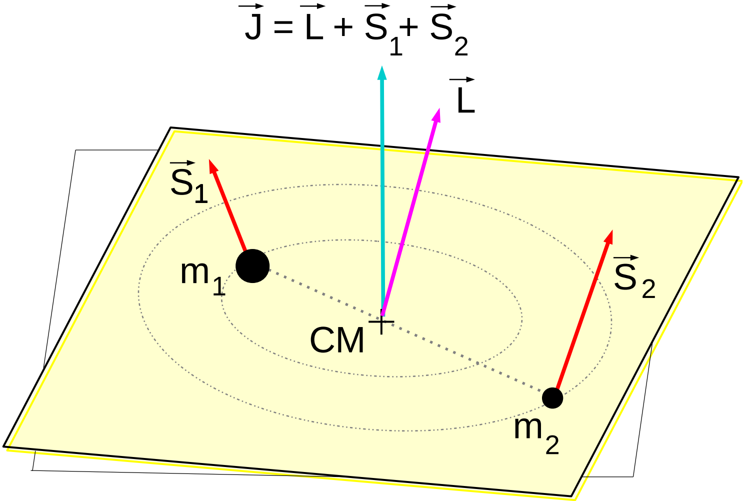
[PRX **6** (2016) 041015]

Spins of supermassive black holes



[CQG **30** (2013) 244004]

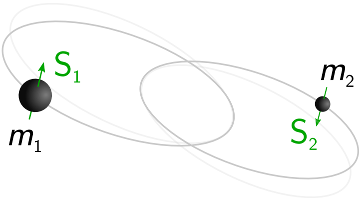
Binary systems of spinning compact bodies



(Figure credit: L. Blanchet)

Spin-orbit coupling at leading order

[Barker & O'Connell 1975]



$$\frac{\mathrm{d}\mathbf{S}_a}{\mathrm{d}t} = \mathbf{\Omega}_a \times \mathbf{S}_a$$

$$H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = H_{\text{orb}}(\mathbf{x}_a, \mathbf{p}_a) + \overbrace{\sum_b \mathbf{\Omega}_b(\mathbf{x}_a, \mathbf{p}_a) \cdot \mathbf{S}_b}^{\text{spin-orbit coupling}}$$

$$\mathbf{\Omega}_1(\mathbf{x}_a, \mathbf{p}_a) = \frac{G}{c^2 r_{12}^2} \left(\frac{3m_2}{2m_1} \mathbf{n}_{12} \times \mathbf{p}_1 - 2\mathbf{n}_{12} \times \mathbf{p}_2 \right) \propto \mathbf{L}$$

Spin effects in the conservative dynamics

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
spin^0	N	1PN		2PN		3PN		4PN				
spin^1			LO SO		NLO SO		NNLO SO					
spin^2				LO S^2		NLO S^2		NNLO S^2				
spin^3							LO S^3		NLO S^3			
spin^4								LO S^4		NLO S^4		
spin^5											LO S^5	
spin^6												LO S^6

[PRD **97** (2018) 064010]

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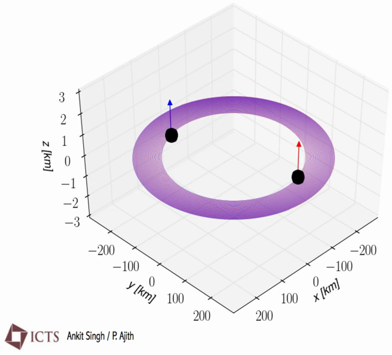
spin^3							LO S^3		NLO S^3			
spin^4								LO S^4		NLO S^4		
spin^5											LO S^5	
spin^6												LO S^6

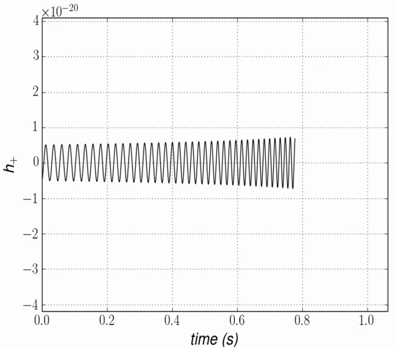
$$H_{\mathrm{LO}}^{\mathrm{BBH}}(m_1, \boldsymbol{a}_1, m_2, \boldsymbol{a}_2) = H_{\mathrm{LO}}^{\mathrm{BBH,test}}(M, \boldsymbol{\sigma}, \mu, \boldsymbol{\sigma}^*)$$

[PRD **97** (2018) 064010]

Spin effects on the waveform

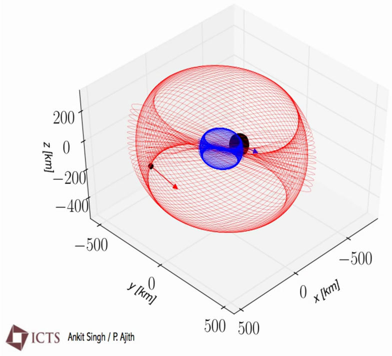
Equal masses and aligned spins

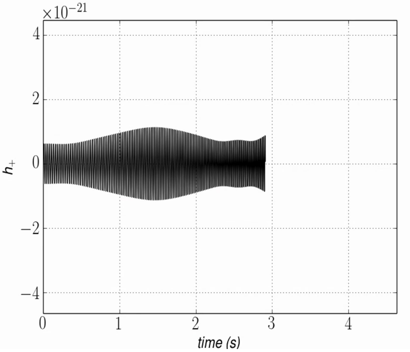




Spin effects on the waveform

Unequal masses and misaligned spins



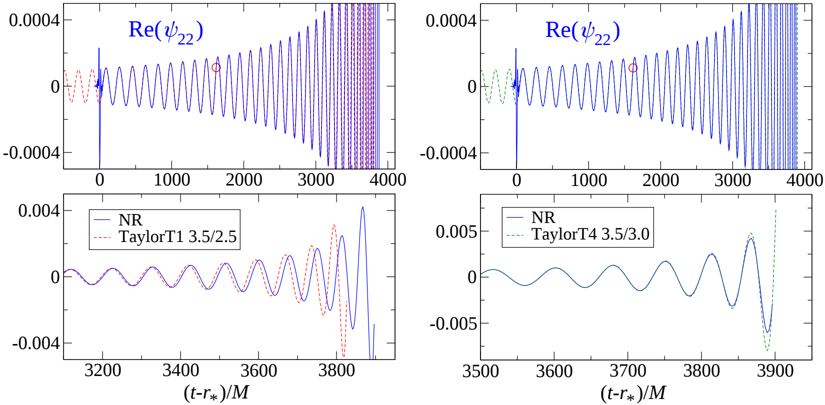


State of the art

	Spinless	Spin-Orbit	Spin-Squared	Tidal
Conserv. dynamics	4PN	3.5PN	3PN	7PN
Energy flux	3.5PN	4PN	2PN	6PN
Radiation reaction	4.5PN	4PN	4.5PN	6PN
Waveform phase	3.5PN	4PN	2PN	6PN
Waveform amplitude	3PN	2PN	2PN	6PN

PN *vs* NR waveforms

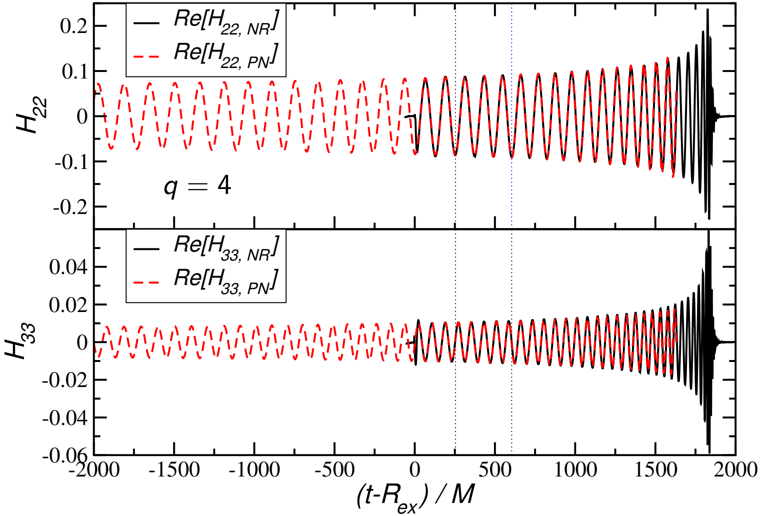
Equal masses and no spins



[PRD **76** (2007) 124038]

PN *vs* NR waveforms

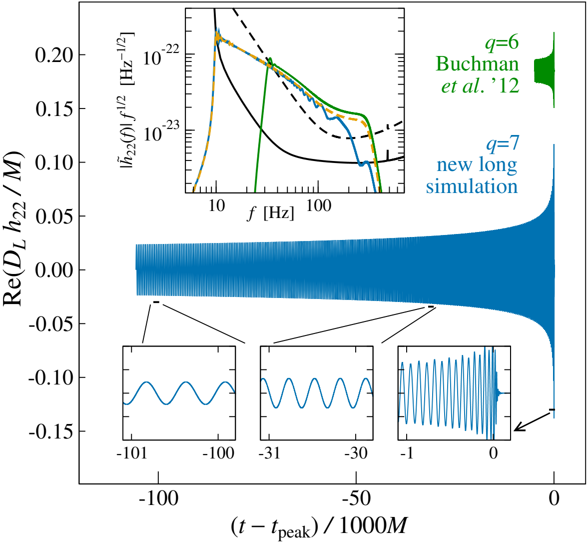
Unequal masses and no spins



[CQG **28** (2011) 134004]

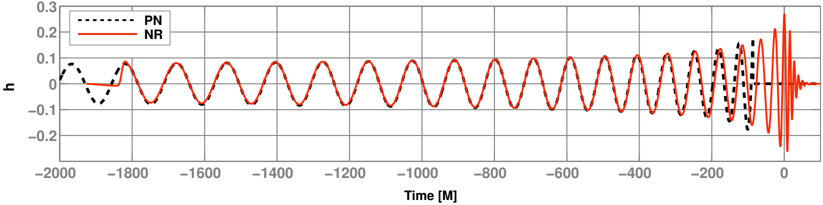
350 GW cycles!

PN *vs* NR waveforms



[PRL **115** (2015) 031102]

Hybrid PN/NR waveforms



[PRD **77** (2008) 104017]

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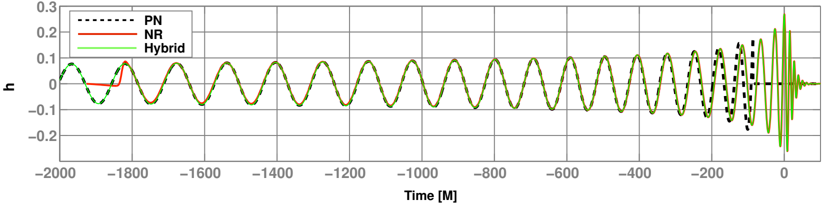


[PRD **77** (2008) 104017]

Further reading

Review articles

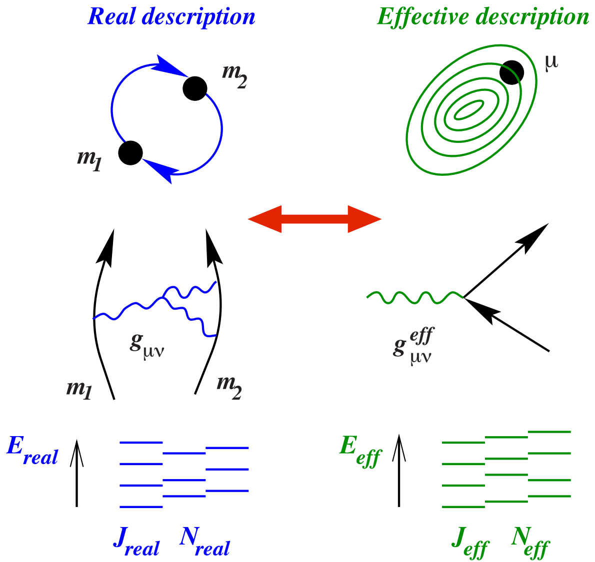
- *Gravitational radiation from post-Newtonian sources...*
  L. Blanchet, Living Rev. Rel. **17**, 2 (2014)
- *Post-Newtonian methods: Analytic results on the binary problem*
  G. Schäfer, in *Mass and motion in general relativity*
  Edited by L. Blanchet et al., Springer (2011)
- *The post-Newtonian approximation for relativistic compact binaries*
  T. Futamase and Y. Itoh, Living Rev. Rel. **10**, 2 (2007)

Topical books

- *Gravity: Newtonian, post-Newtonian, relativistic*
  E. Poisson and C. M. Will, Cambridge University Press (2015)
- *Gravitational waves: Theory and experiments*
  M. Maggiore, Oxford University Press (2007)

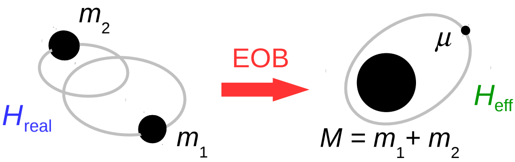
Outline

1. Gravitational wave source modelling
2. Post-Newtonian approximation
3. Effective one-body model

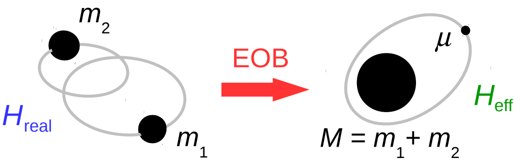


$$E_{\text{eff}}(J, N) = f(E_{\text{real}}(J, N))$$

(Credit: Buonanno & Sathyaprakash 2015)



- Motivated by the exact solution in the Newtonian limit



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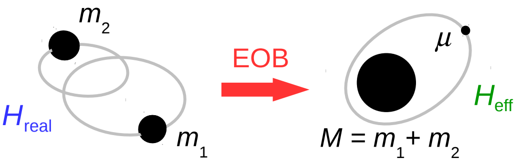

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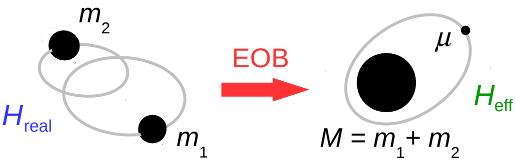

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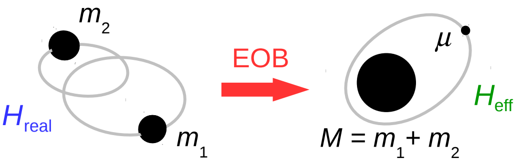

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- Idea extended to spinning binaries and to tidal effects

EOB Hamiltonian dynamics

EOB Hamiltonian

$$H_{\text{real}}^{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}, \quad \nu \equiv \frac{\mu}{M} \in [0, 1/4]$$

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Effective Hamiltonian

$$H_{\rm eff} = \mu\sqrt{g_{tt}^{\rm eff}(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{g_{rr}^{\rm eff}(r)} + \cdots\right)}$$

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Hamilton's equations

$$\dot{r} = \frac{\partial H_{\rm real}^{\rm EOB}}{\partial p_r}\,, \quad \dot{p}_r = -\frac{\partial H_{\rm real}^{\rm EOB}}{\partial r} + F_r\,, \quad \cdots$$

EOB effective metric

Effective metric

$$\mathrm{d}s_{\text{eff}}^2 = -g_{tt}^{\text{eff}}(r;\nu)\,\mathrm{d}t^2 + g_{rr}^{\text{eff}}(r;\nu)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

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Effective potentials

$$g_{tt}^{\text{eff}} = \underbrace{1 - \frac{2M}{r}}_{\text{Schwarzschild}} + \underbrace{\nu\left[2\left(\frac{M}{r}\right)^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\left(\frac{M}{r}\right)^4 + \cdots\right]}_{\text{finite mass-ratio ``deformation''}}$$

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Padé resummation

Motivation: improve convergence of PN series in strong-field regime

EOB waveform generation

Inspiral/plunge

Evolution of Hamiltonian dynamics up to EOB light-ring
Resummations of PN waveform modes and fluxes

$$h^{\text{inspiral}}(t) = \text{``big mess''}$$

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Merger/ringdown

Impose continuity with black hole quasinormal modes ringing

$$h^{\text{ringdown}}(t) = \sum_{n\ell m} C_{n\ell m}\, e^{-t/\tau_{n\ell m}} \cos\left(\omega_{n\ell m}(t - t_{\mathrm{m}})\right)$$

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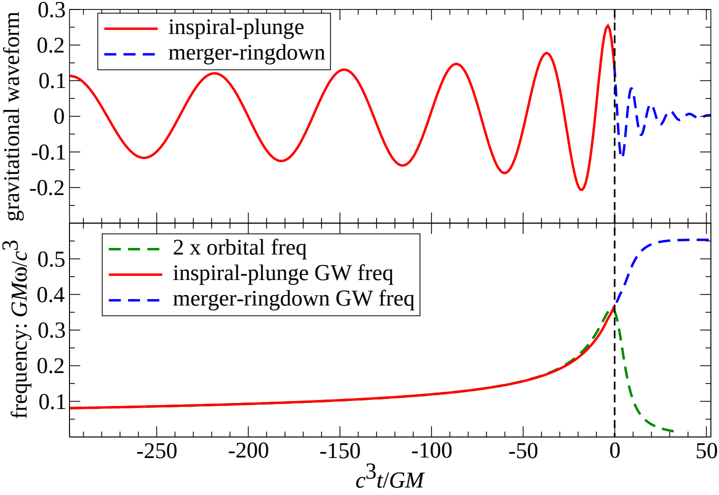
Final EOB waveform

$$h^{\text{EOB}}(t) = \Theta(t_{\text{m}} - t)\, h^{\text{inspiral}}(t) + \Theta(t - t_{\text{m}})\, h^{\text{ringdown}}(t)$$

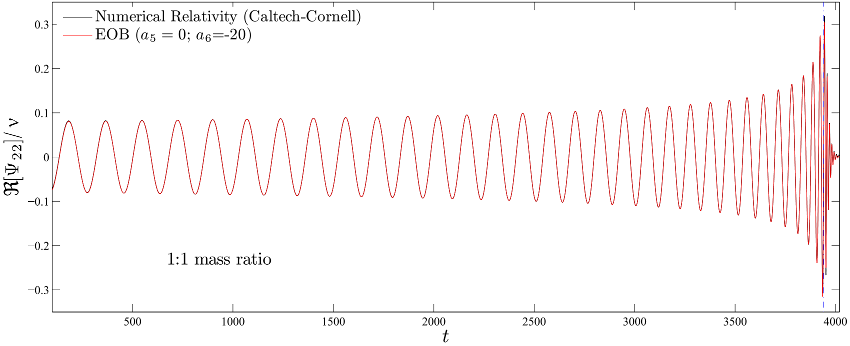
EOB waveform prediction



(Credit: Buonanno & Sathyaprakash 2015)

EOB *vs* NR waveforms

Equal masses and no spins



[PRD **79** (2009) 081503]

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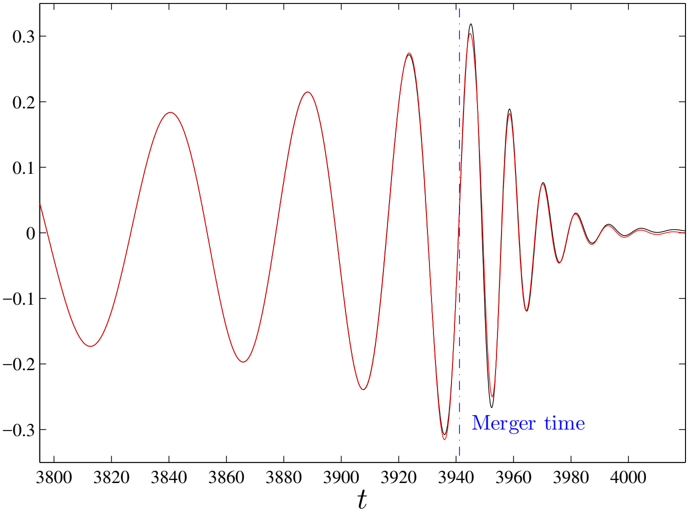

[PRD **79** (2009) 081503]

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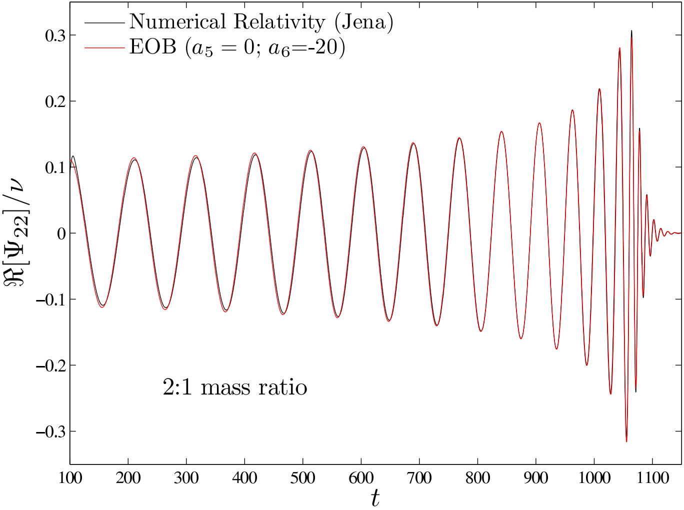

[PRD **79** (2009) 081503]

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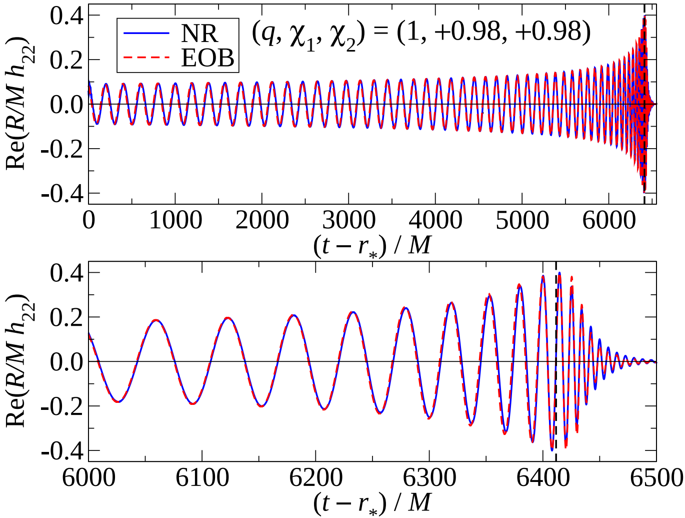
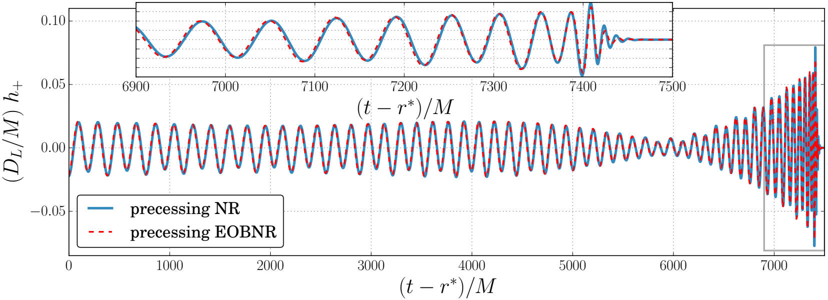


[PRD **89** (2014) 061502]

EOB *vs* NR waveforms

Unequal masses and a precessing spin

$$(q, \chi_1, \chi_2) = (5, +0.5, 0),\ \iota = \pi/3$$



[PRD **95** (2017) 024010]

Recent developments

- Extension of EOB model to **spinning** binaries

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- Various **calibrations** to numerical relativity simulations
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- Calibration of EOB potentials by comparison to **self-force**
  [Barack et al. (2011), Le Tiec (2015), Akcay & van de Meent (2016)]

Further reading

Review articles

- *Sources of gravitational waves: Theory and observations*
  A. Buonanno and B. S. Sathyaprakash, in *General relativity and gravitation: A centennial perspective*
  Edited by A. Ashtekar et al., Cambridge University Press (2015)

- *The general relativistic two body problem and the EOB formalism*
  T. Damour, in *General relativity, cosmology and astrophysics*
  Edited by J. Bicák and T. Ledvinka, Springer (2014)

- *The effective one-body description of the two-body problem*
  T. Damour and A. Nagar, in *Mass and motion in general relativity*
  Edited by L. Blanchet et al., Springer (2011)