

The shape of black hole Love

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TIDES

Low tide

High tide



High tide



Moon

Low tide

① Newtonian tides

② Relativistic tides

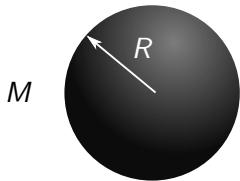
③ Black hole Love numbers

① Newtonian tides

② Relativistic tides

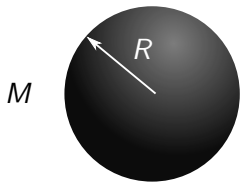
③ Black hole Love numbers


Newtonian theory of Love numbers



$$U = \frac{M}{r}$$

Newtonian theory of Love numbers



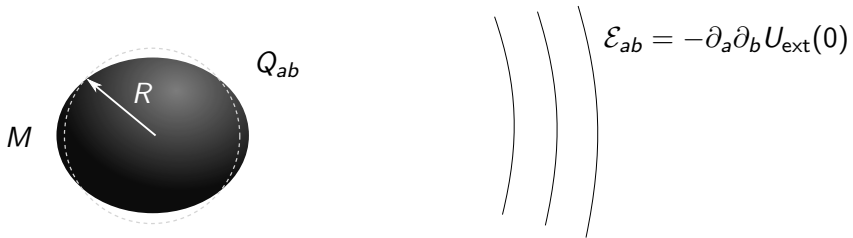


Three vertical, slightly curved lines representing external potential. To the right of these lines is the equation $\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(0)$.

$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(0)$$

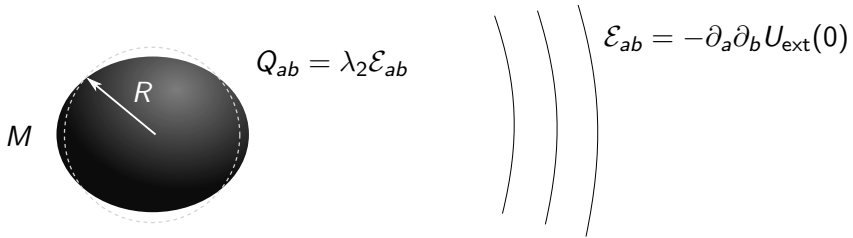
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab}$$

Newtonian theory of Love numbers



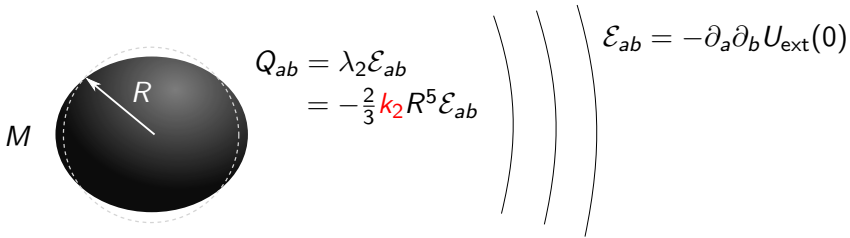
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} + \frac{3}{2} \frac{x^a x^b Q_{ab}}{r^5}$$

Newtonian theory of Love numbers



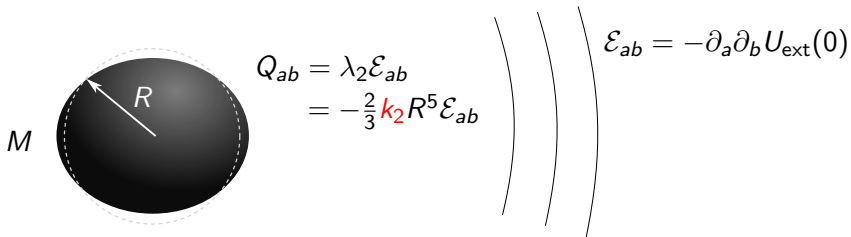
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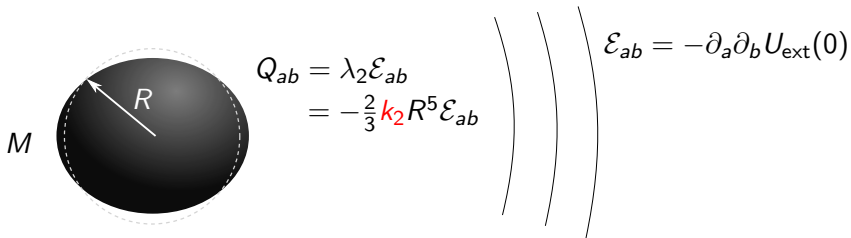
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Newtonian theory of Love numbers



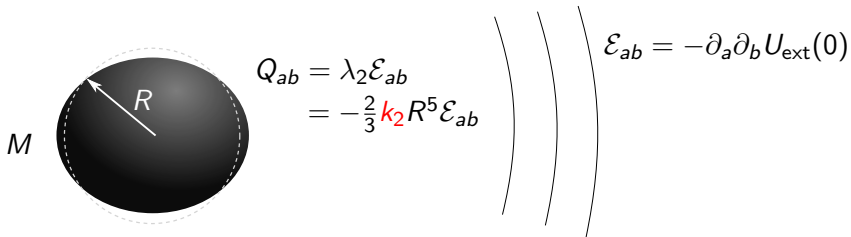
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} \left[1 + 2k_2 \left(\frac{R}{r} \right)^5 \right]$$

Newtonian theory of Love numbers



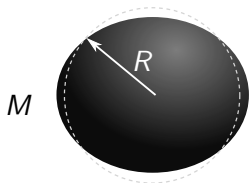
$$U = \frac{M}{r} - \sum_{\ell \geq 2} \frac{(\ell-2)!}{\ell!} x^{a_1} \dots x^{a_\ell} \mathcal{E}_{a_1 \dots a_\ell} \left[1 + 2k_\ell \left(\frac{R}{r} \right)^{2\ell+1} \right]$$

Newtonian theory of Love numbers



$$U = \frac{M}{r} - \sum_{\ell \geq 2} \sum_{|m| \leq \ell} \frac{(\ell-2)!}{\ell!} r^\ell \mathcal{E}_{\ell m} \left[1 + 2k_\ell \left(\frac{R}{r} \right)^{2\ell+1} \right] Y_{\ell m}$$

Newtonian theory of Love numbers

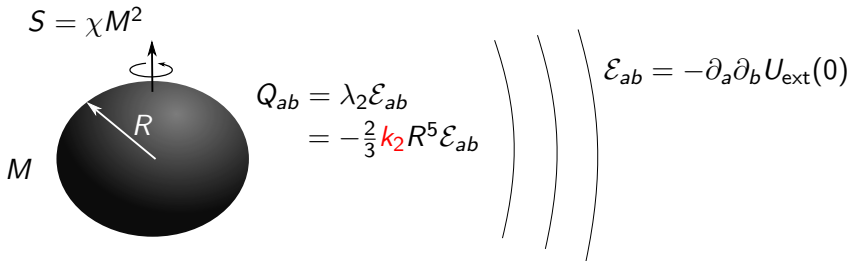


$$\begin{aligned} Q_{ab} &= \lambda_2 \mathcal{E}_{ab} \\ &= -\frac{2}{3} k_2 R^5 \mathcal{E}_{ab} \end{aligned}$$

$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(0)$$

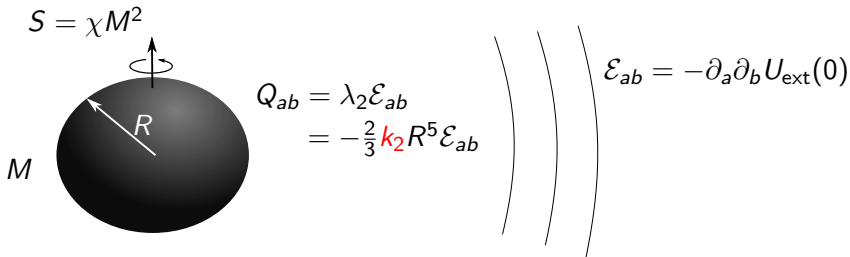
$$\psi_0 = \sum_{\ell \geq 2} \sum_{|m| \leq \ell} \sqrt{\frac{(\ell+2)(\ell+1)}{\ell(\ell-1)}} r^{\ell-2} \mathcal{E}_{\ell m} \left[1 + 2k_\ell \left(\frac{R}{r} \right)^{2\ell+1} \right] {}_2Y_{\ell m}$$

Newtonian theory of Love numbers



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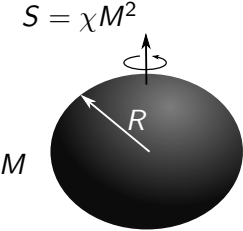
Newtonian theory of Love numbers



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$$k_{\ell m} = k_{\ell}^{(0)} + im\chi k_{\ell}^{(1)} + O(\chi^2)$$

Newtonian theory of Love numbers



$S = \chi M^2$
 M
 R

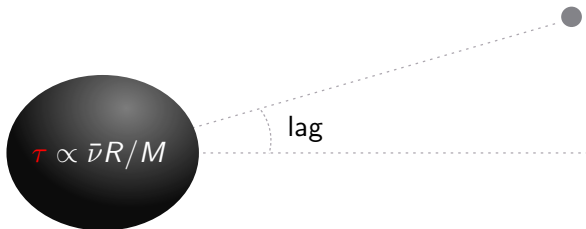
$$Q_{ab} = \lambda_2 \mathcal{E}_{ab} = -\frac{2}{3} k_2 R^5 \mathcal{E}_{ab}$$

$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(0)$$

$$\psi_0 = \sum_{\ell \geq 2} \sum_{|m| \leq \ell} \sqrt{\frac{(\ell+2)(\ell+1)}{\ell(\ell-1)}} r^{\ell-2} \mathcal{E}_{\ell m} \left[1 + 2k_{\ell m} \left(\frac{R}{r} \right)^{2\ell+1} \right] {}_2Y_{\ell m}$$

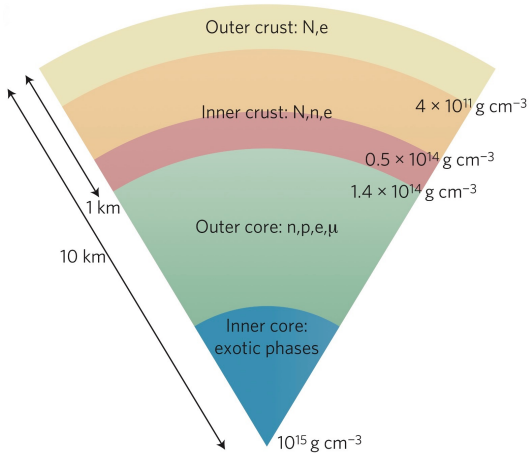
Tidal Love numbers $k_{\ell m}$ \longleftrightarrow **body's internal structure**

Tidal dissipation: lag, heating and torquing



$$\begin{aligned} Q_{ab}(t) &= -\frac{2}{3} k_2 R^5 [\mathcal{E}_{ab}(t) - \tau \dot{\mathcal{E}}_{ab}(t) + \dots] \\ &= -\frac{2}{3} k_2 R^5 [\mathcal{E}_{ab}(t - \tau) + \dots] \end{aligned}$$

Internal structure of neutron stars

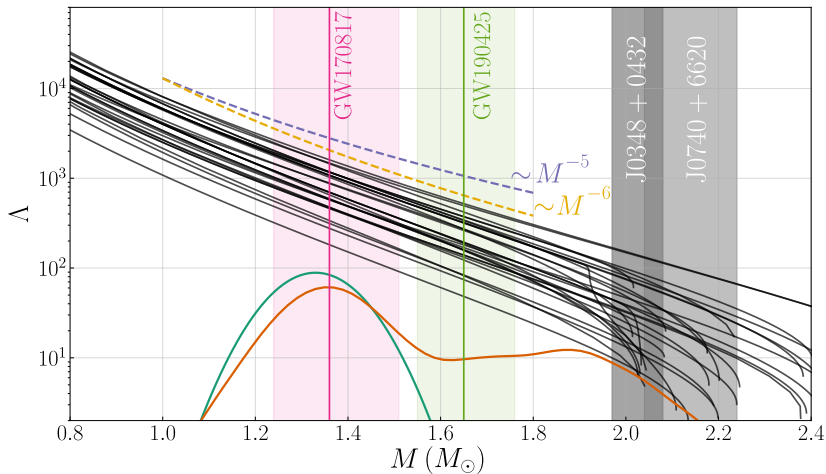


GW observations as probes of **neutron star internal structure**

Gravitational-wave observations

[Chatziioannou, GRG 2020]

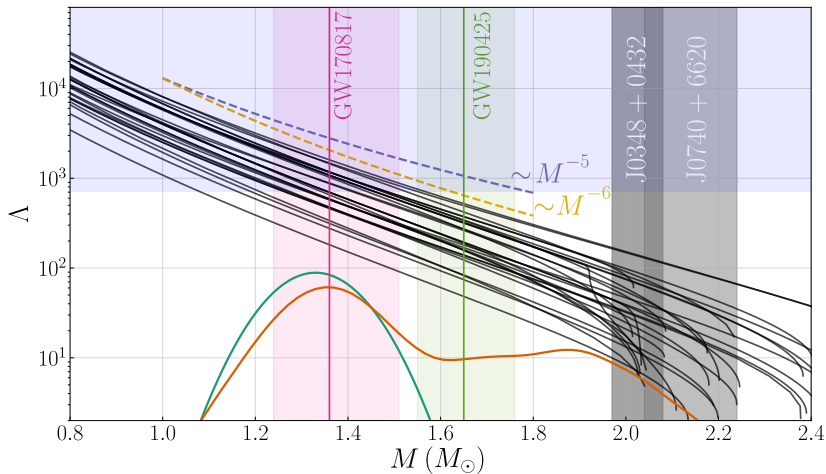
$$\Lambda \equiv \lambda_2/M^5 \propto k_2/C^5$$



Gravitational-wave observations

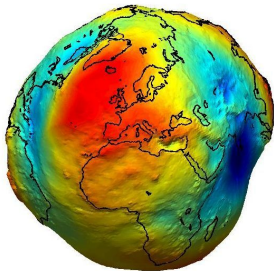
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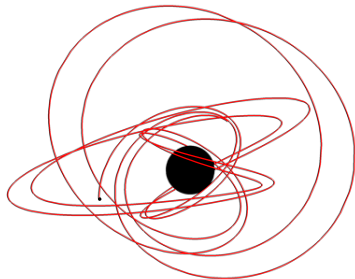
Do black holes have hair?

Geodesy



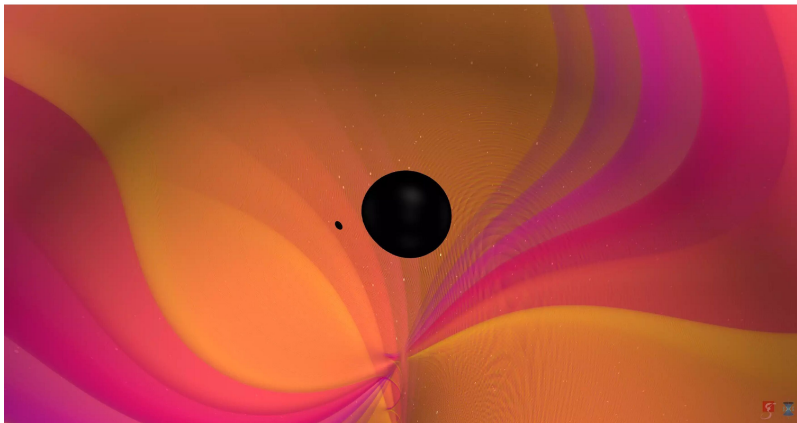
$M_{\ell m}$ arbitrary

Botromeladesy



$$M_{\ell 0} + iS_{\ell 0} = M(ia)^\ell$$

Do black holes tidally deform?



Related to Ashtekar's *gravitational-wave tomography* research program

① Newtonian tides

② Relativistic tides

③ Black hole Love numbers

Relativistic theory of Love numbers

- Electric-type and magnetic-type tidal moments:

$$\mathcal{E}_L \propto [C_{0a_1 0a_2; a_3 \dots a_\ell}]^{\text{STF}} \quad \text{and} \quad \mathcal{B}_L \propto [\varepsilon_{a_1 bc} C_{a_2 0bc; a_3 \dots a_\ell}]^{\text{STF}}$$

Relativistic theory of Love numbers

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- Metric and Geroch-Hansen multipole moments:

$$g_{\alpha\beta} = \dot{g}_{\alpha\beta} + \underbrace{h_{\alpha\beta}^{\text{tidal}}}_{\sim r^\ell} + \underbrace{h_{\alpha\beta}^{\text{resp}}}_{\sim r^{-(\ell+1)}} \quad \longrightarrow \quad \begin{cases} M_L = \dot{M}_L + \delta M_L \\ S_L = \dot{S}_L + \delta S_L \end{cases}$$

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- Two families of tidal deformability parameters:

$$\delta M_L = \lambda_\ell^{\text{el}} \mathcal{E}_L \quad \text{and} \quad \delta S_L = \lambda_\ell^{\text{mag}} \mathcal{B}_L$$

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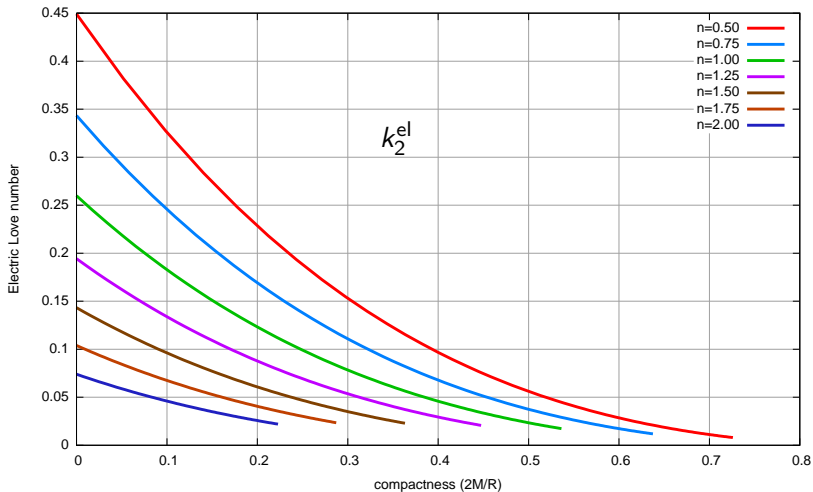
$$\delta M_L = \lambda_\ell^{\text{el}} \mathcal{E}_L \quad \text{and} \quad \delta S_L = \lambda_\ell^{\text{mag}} \mathcal{B}_L$$

- Dimensionless tidal Love numbers:

$$k_\ell^{\text{el/mag}} \equiv -\frac{(2\ell - 1)!!}{2(\ell - 2)!} \frac{\lambda_\ell^{\text{el/mag}}}{R^{2\ell+1}}$$

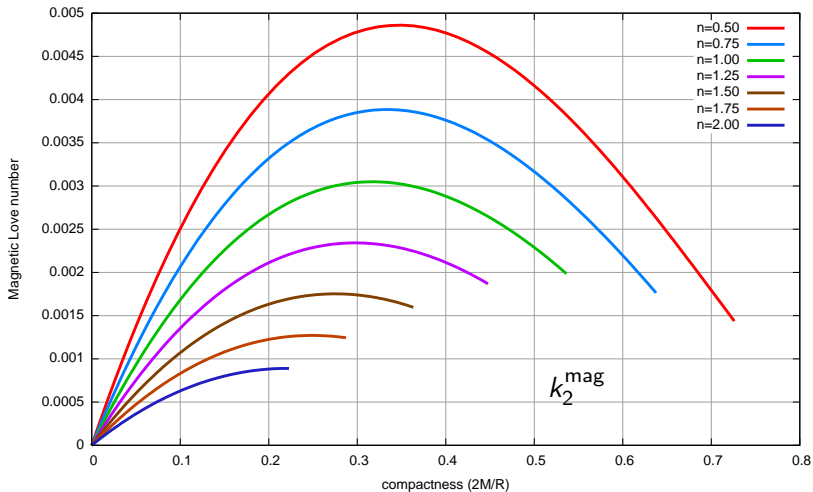
Love numbers of neutron stars

[Binnington & Poisson, PRD 2009]



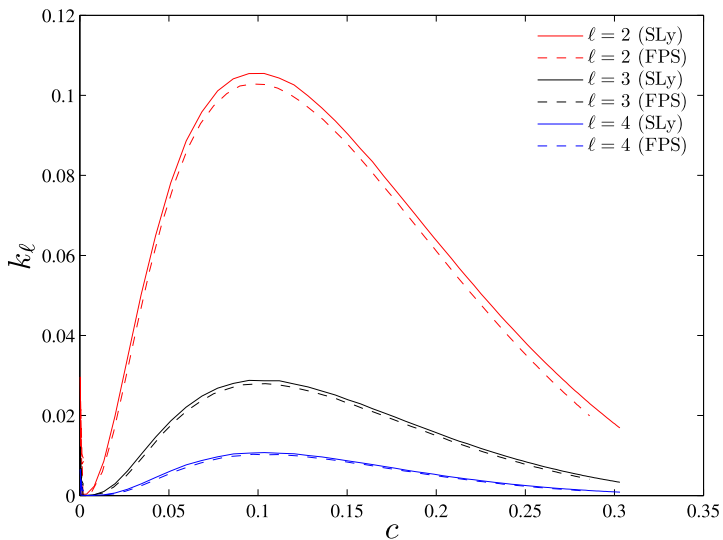
Love numbers of neutron stars

[Binnington & Poisson, PRD 2009]



Love numbers of neutron stars

[Damour & Nagar, PRD 2009]



Love numbers of *spinning* compact objects

- The spin breaks the spherical symmetry of the background
 - No proportionality between $(\delta M_L, \delta S_L)$ and $(\mathcal{E}_L, \mathcal{B}_L)$
 - Degeneracy of the azimuthal number m lifted
 - Parity mixing and mode couplings allowed

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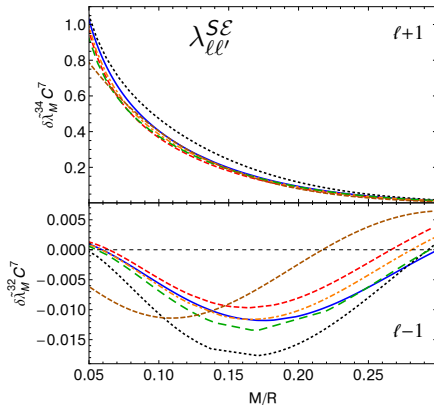
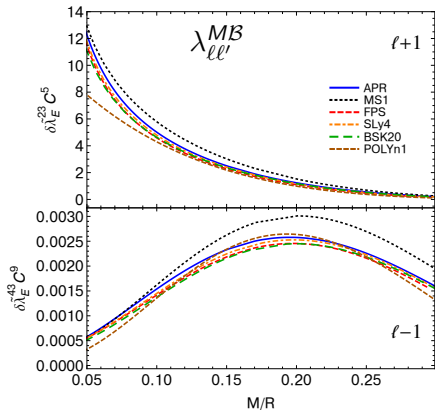
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- *Four* families of tidal deformability parameters:

$$\begin{aligned} \lambda_{\ell\ell'm}^{M\mathcal{E}} &\equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{E}_{\ell'm}} & \lambda_{\ell\ell'm}^{S\mathcal{B}} &\equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{B}_{\ell'm}} \\ \lambda_{\ell\ell'm}^{S\mathcal{E}} &\equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{E}_{\ell'm}} & \lambda_{\ell\ell'm}^{M\mathcal{B}} &\equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{B}_{\ell'm}} \end{aligned}$$

Love numbers of *spinning* neutron stars

[Pani, Gualtieri & Ferrari, PRD 2015]



Restricted to an *axisymmetric* tidal perturbation ($m = 0$)

① Newtonian tides

② Relativistic tides

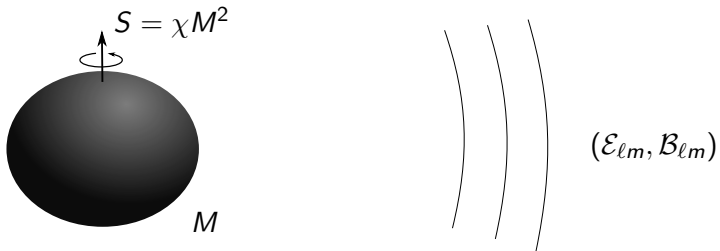
③ Black hole Love numbers

Do black holes have zero Love numbers?

Reference	Background	Tidal field
[Binnington & Poisson 2009]	Schwarzschild	weak, generic ℓ
[Damour & Nagar 2009]	Schwarzschild	weak, generic ℓ
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $\ell = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(S^2)$	weak, $(\ell, m) = (2, 0)$
[Le Tiec & Casals 2021]	Exact Kerr	weak, generic ℓ
[Chia 2021]	Exact Kerr	weak, generic ℓ
[Goldberger et al. 2021]	Exact Kerr	weak, generic ℓ
[Charalambous et al. 2021]	Exact Kerr	weak, generic ℓ
[Hui 2021]	Schwar-(A)dS	weak, generic ℓ

Problem of **fine-tuning** from an Effective-Field-Theory perspective?

Investigating Kerr's Love



$$(\mathcal{E}_{\ell m}, \mathcal{B}_{\ell m}) \rightarrow \psi_0 \rightarrow \Psi \rightarrow h_{\alpha\beta} \rightarrow (M_{\ell m}, S_{\ell m}) \rightarrow \lambda_{\ell m}^{M/S, \mathcal{E}/\mathcal{B}}$$

Metric reconstruction through the Hertz potential Ψ

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c \rightarrow \infty} \psi_0^{\ell m} \propto \mathcal{E}_{\ell m} r^{\ell-2} [1 + 2k_{\ell m} (R/r)^{2\ell+1}] {}_2Y_{\ell m}(\theta, \phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{\ell m} \propto [\mathcal{E}_{\ell m} + \frac{3i}{\ell+1} \mathcal{B}_{\ell m}] R_{\ell m}(r) {}_2Y_{\ell m}(\theta, \phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{\ell m}(r) = \underbrace{r^{\ell-2} (1 + \dots)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + k_{\ell m} \underbrace{r^{-\ell-3} (1 + \dots)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Why analytic continuation?

$$R_{\ell m}(r) = \underbrace{r^{\ell-2} (1 + \dots)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + k_{\ell m} \underbrace{r^{-\ell-3} (1 + \dots)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Ambiguity in the linear response [Fang & Lovelace 2005; Gralla 2018]

The decaying solution $R_{\ell m}^{\text{resp}}$ is affected by a radial coord. transfo.

Ambiguity in the tidal field [Pani, Gualtieri, Maselli & Ferrari 2015]

The growing solution $R_{\ell m}^{\text{tidal}} + \alpha R_{\ell m}^{\text{resp}}$ still qualifies as a tidal solution

Kerr black hole linear response

$$R_{\ell m}(r) = \underbrace{R_{\ell m}^{\text{tidal}}(r)}_{\sim r^{\ell-2}} + 2k_{\ell m} \underbrace{R_{\ell m}^{\text{resp}}(r)}_{\sim r^{-(\ell+3)}}$$

- The coefficients $k_{\ell m}$ can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$$k_{\ell m} = -im\chi \frac{(\ell+2)!(\ell-2)!}{4(2\ell+1)!(2\ell)!} \prod_{n=1}^{\ell} [n^2(1-\chi^2) + m^2\chi^2]$$

- The linear response vanishes identically when:
 - the black hole spin vanishes ($\chi = 0$)
 - the tidal field is axisymmetric ($m = 0$)
- Reconstruct the Kerr black hole response $h_{\alpha\beta}^{\text{resp}}$ via Ψ^{resp}

Love numbers of a Kerr black hole

- We compute the Love numbers to **linear** order in $\chi \equiv S/M^2$

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- The modes of the mass/current **quadrupole** moments are

$$\delta M_{2m} \doteq \frac{im\chi}{180} (2M)^5 \mathcal{E}_{2m} \quad \text{and} \quad \delta S_{2m} \doteq \frac{im\chi}{180} (2M)^5 \mathcal{B}_{2m}$$

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- The associated dimensionless tidal Love numbers are

$$k_{2m}^{ME} = k_{2m}^{SB} \doteq -\frac{im\chi}{120} \quad \text{and} \quad k_{2m}^{MB} = k_{2m}^{SE} \doteq 0$$

Love numbers of a Kerr black hole

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- The modes of the mass/current **quadrupole** moments are

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- The associated dimensionless tidal Love numbers are

$$k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} \doteq -\frac{im\chi}{120} \quad \text{and} \quad k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} \doteq 0$$

- For a dimensionless black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \simeq 2 \times 10^{-3} \quad \longrightarrow \quad \text{black holes are “rigid”}$$

Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\text{el}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\text{mag}} \mathcal{B}_{ab}$$

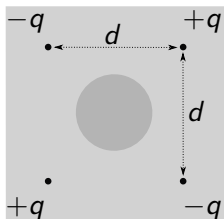
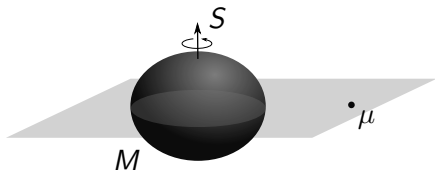
- For a **spinning black hole** we have the more general **tensorial** relations

$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}_{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}_{cd}$$

- To *linear order* in the black hole spin vector S^a we find

$$\delta M_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{E}^d_{(a} \mathcal{E}_{b)cd}$$
$$\delta S_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{B}^d_{(a} \mathcal{B}_{b)cd}$$

Newtonian static quadrupolar tide



$$\mathcal{E}_{ab} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

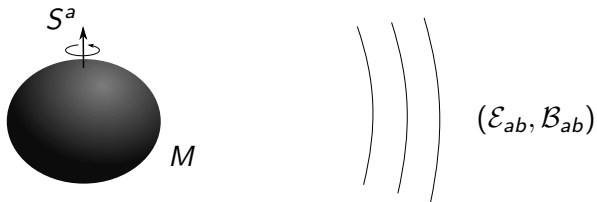
$$\delta M_{ab} \doteq 3Q \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↑

$$\frac{\chi}{180} (2M)^5 \frac{\mu}{r^3} = qd^2$$

Tidal torquing of a spinning black hole

[Thorne & Hartle 1980; Poisson 2004]



- An arbitrary spinning body interacting with a tidal environment suffers a **tidal torquing**:

$$\langle \dot{S}^a \rangle = -\epsilon^{abc} \langle M_{bd} \mathcal{E}^d_c + S_{bd} \mathcal{B}^d_c \rangle$$

- Applied to a **spinning black hole** this yields

$$\langle \dot{S} \rangle \doteq -\frac{8}{45} M^5 \chi [2\langle E_1 + B_1 \rangle - 3\langle E_2 + B_2 \rangle]$$

Summary

- Love numbers of Kerr black holes **do not vanish** in general
- We computed in closed-form the leading (quadrupolar) Love numbers to linear order in the black hole spin
- **Kerr black holes deform** like any other self-gravitating body, despite being particularly “rigid” compact objects
- This is closely related to the phenomenon of **tidal torquing**
- **New black hole test** of the Kerr-like nature of the massive compact objects at the center of galaxies?

Spinning black holes fall in Love!