General relativistic dynamics of binary black holes

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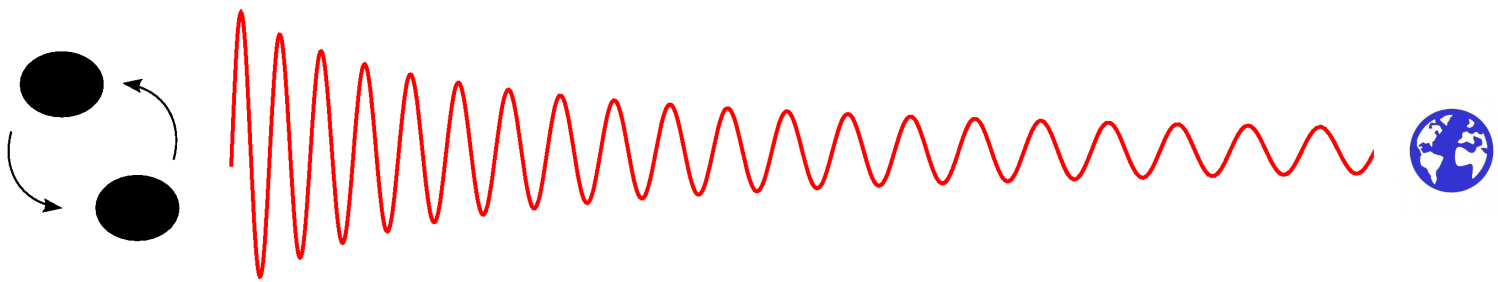

Outline

1. Context
2. Source modelling
3. EMRIs
4. Comparisons

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2. Source modelling
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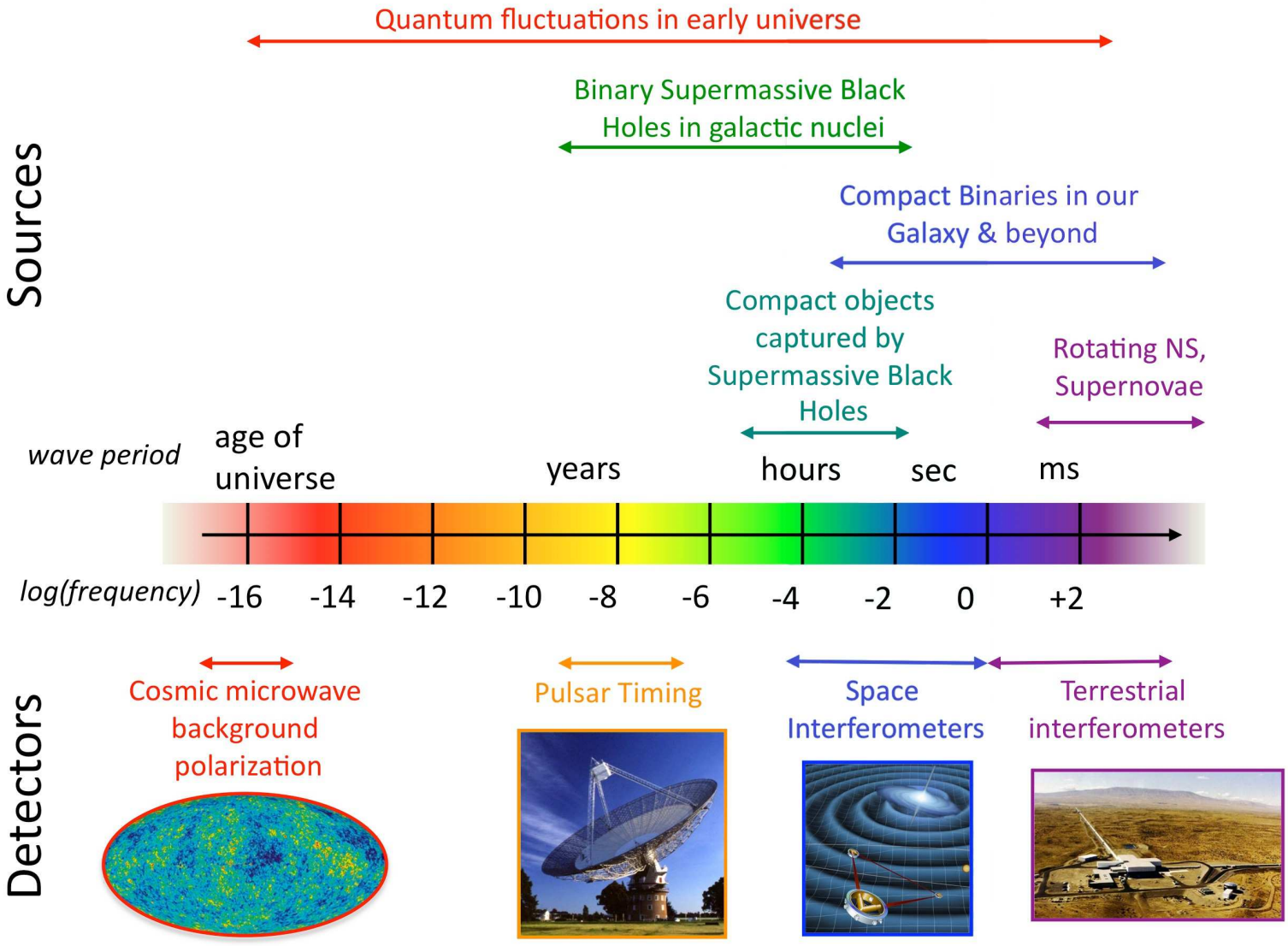
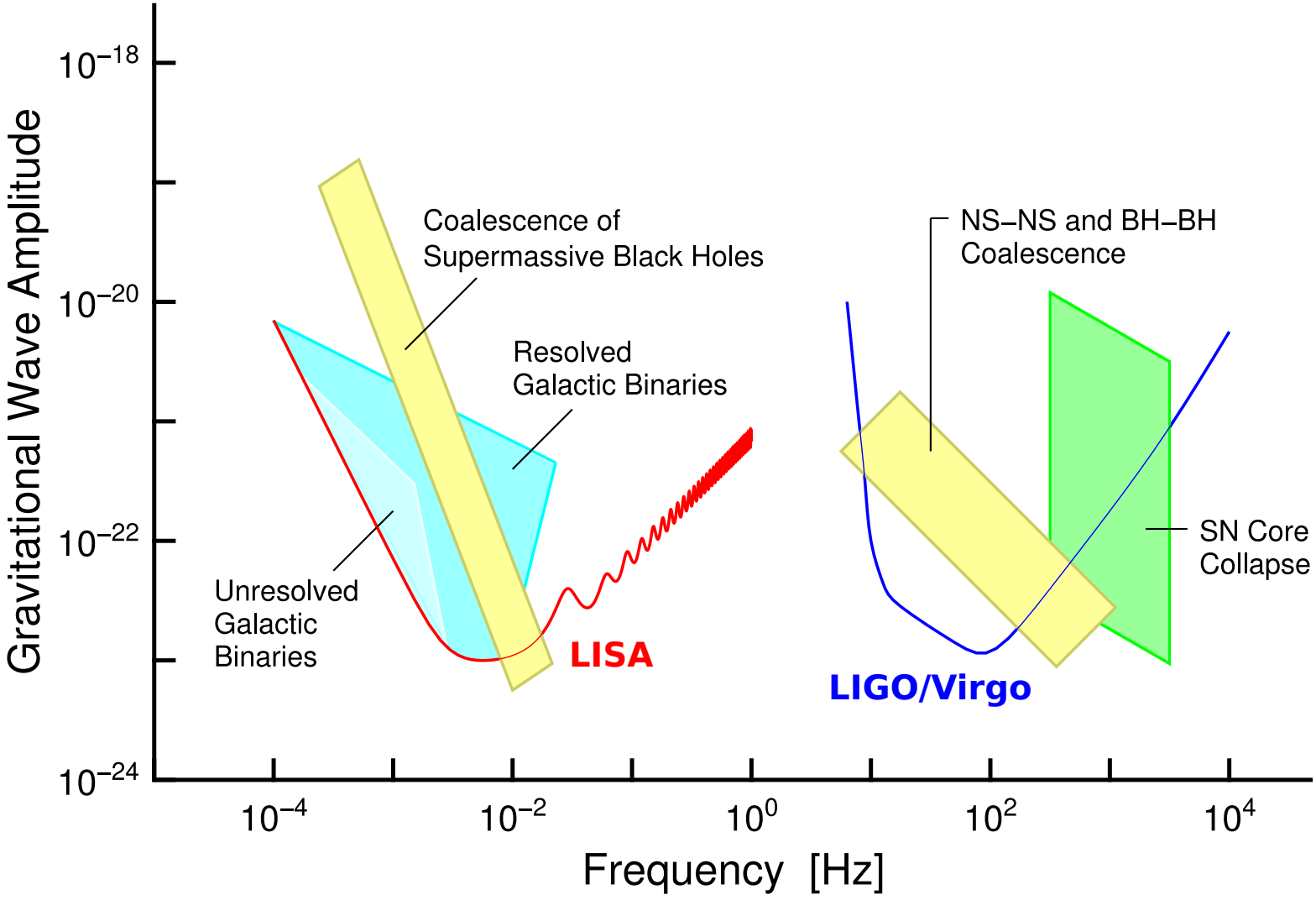
4. Comparisons

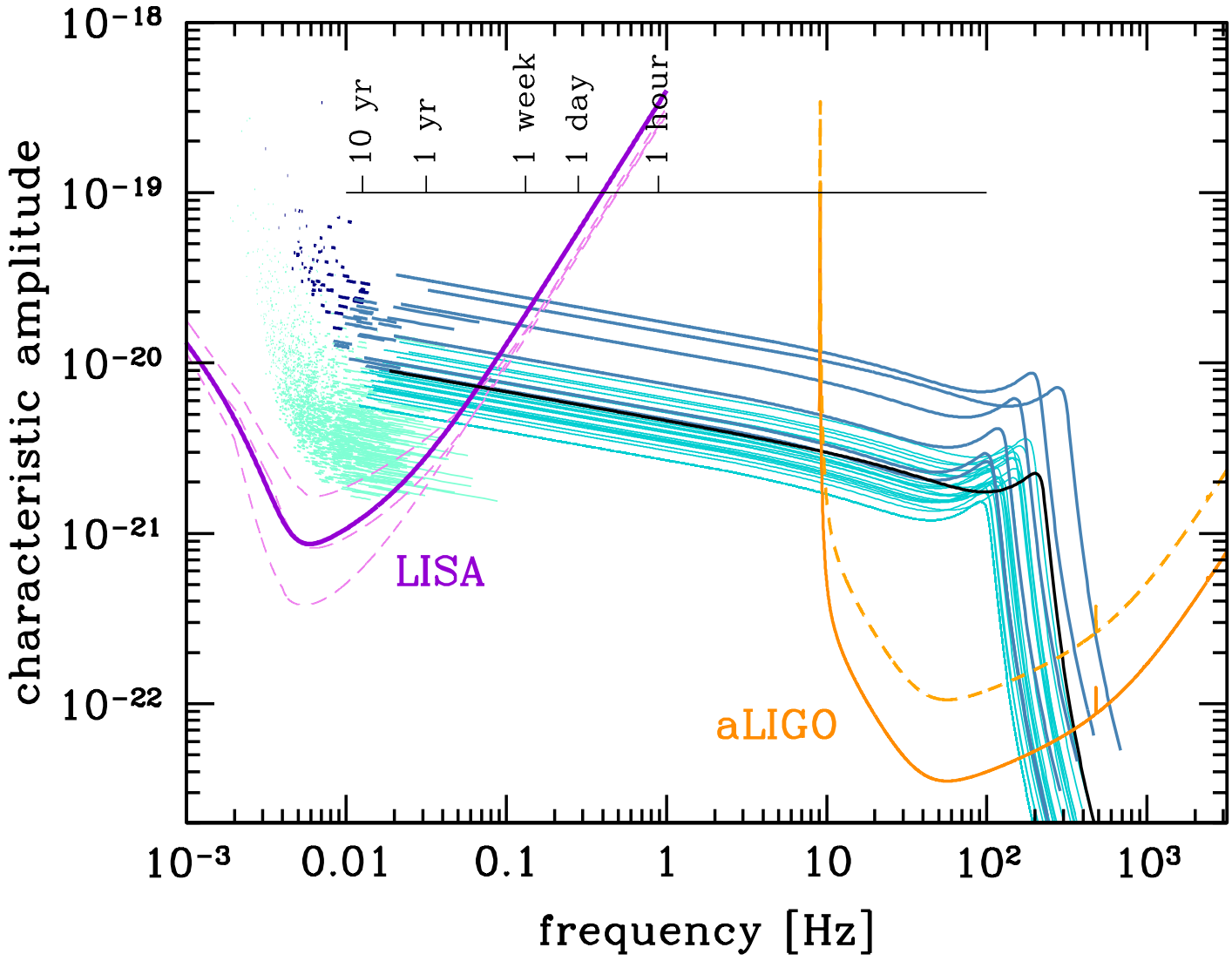
The gravitational-wave spectrum



Promising sources of gravitational waves



Multi-band gravitational-wave astronomy



[PRL **116** (2016) 231102]

Gravitational-wave science

Fundamental physics

- Strong-field tests of GR
- Black hole no-hair theorem
- Cosmic censorship conjecture
- Dark energy equation of state
- Matter at supranuclear densities

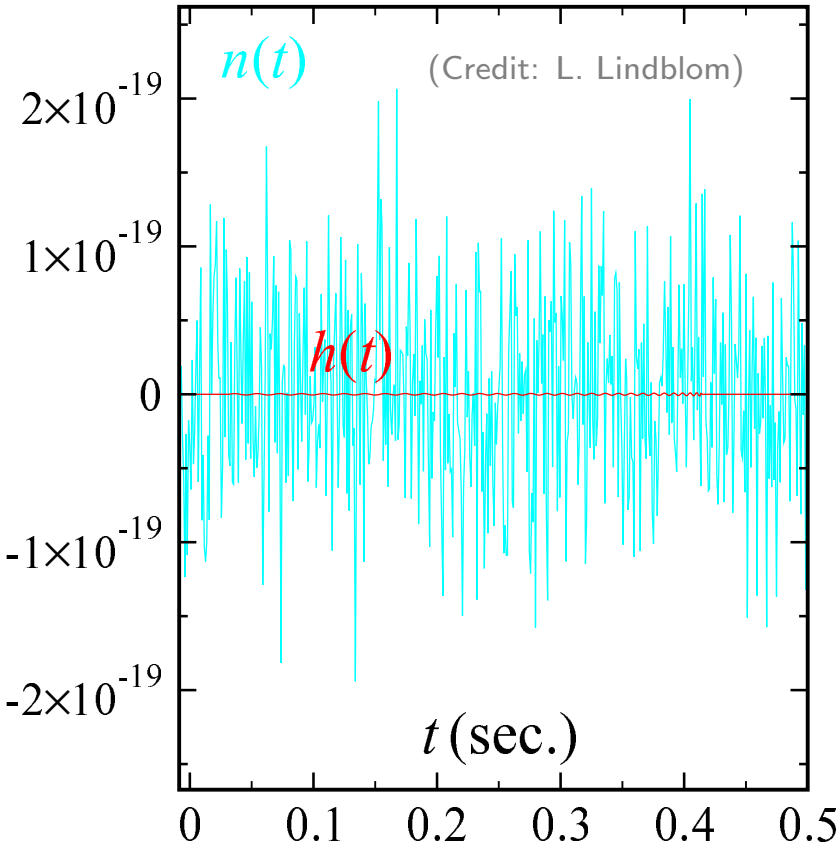
Astrophysics

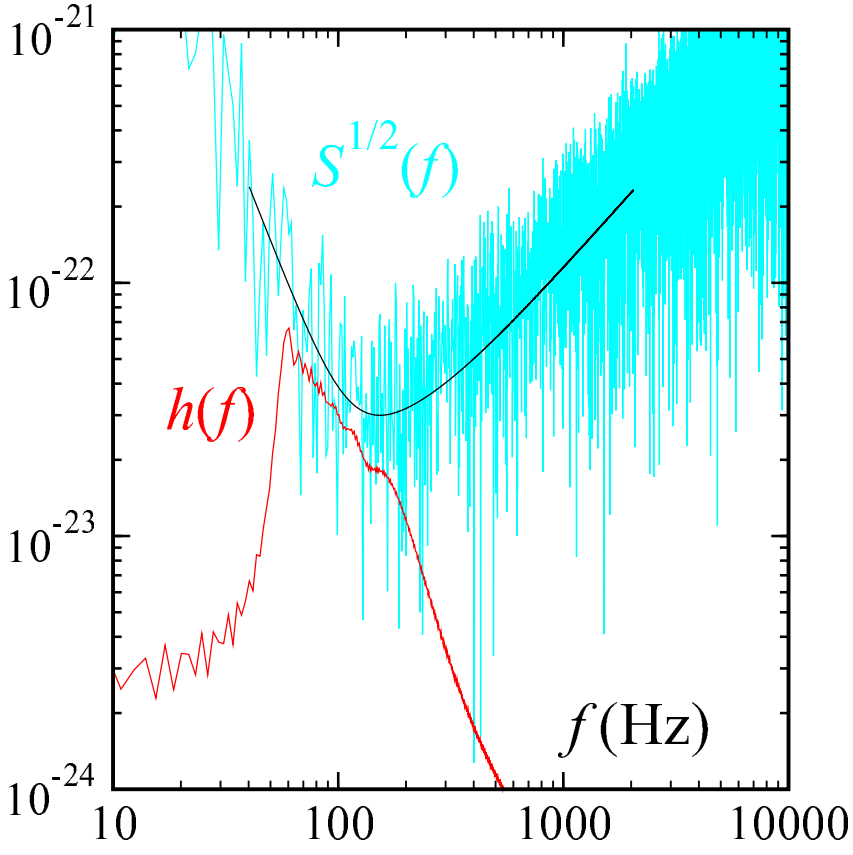
- Formation and evolution of compact binaries
- Origin and mechanisms of γ-ray bursts
- Internal structure of neutron stars

Cosmology

- Cosmography and measure of Hubble's constant
- Origin and growth of supermassive black holes

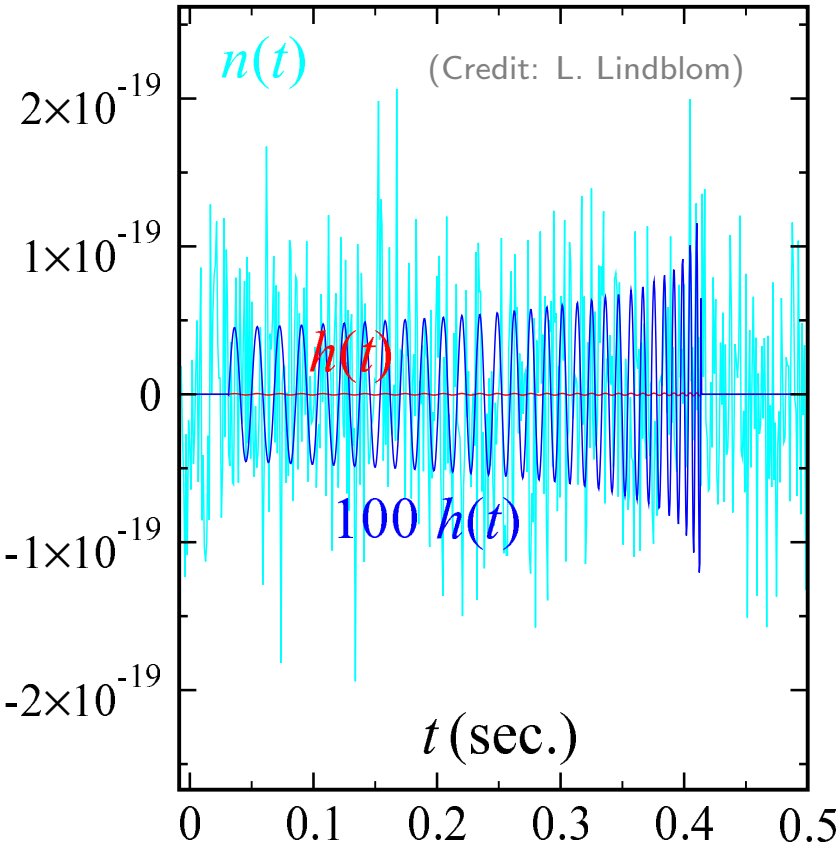
Need for accurate template waveforms

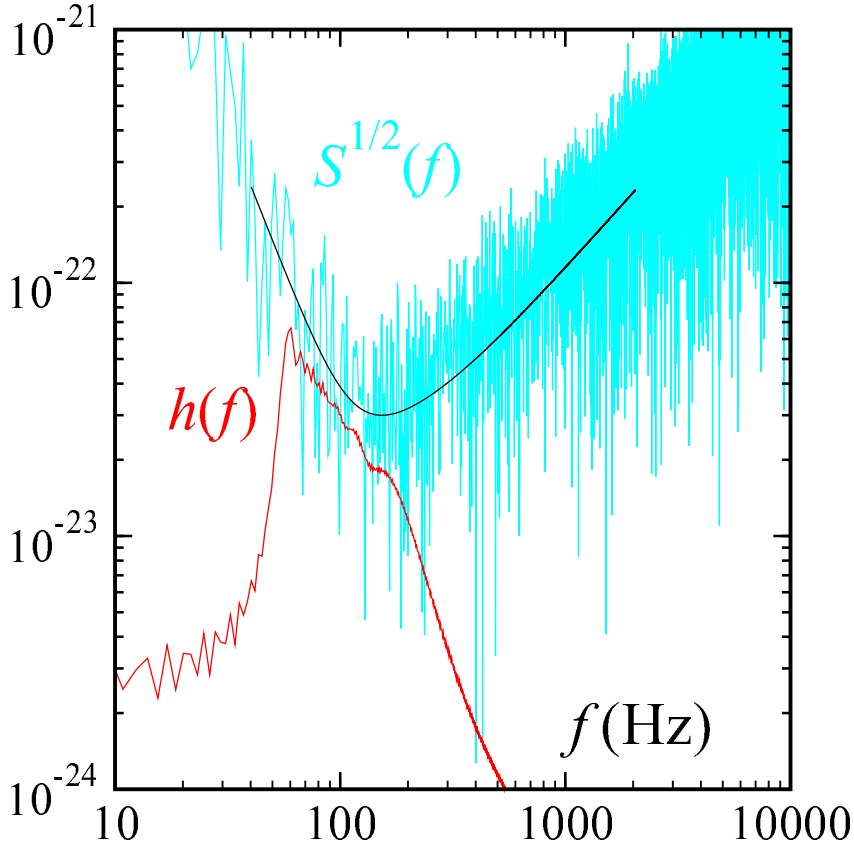




If the expected signal is *known in advance* then $n(t)$ can be filtered and $h(t)$ recovered by matched filtering $\longrightarrow$ **template waveforms**

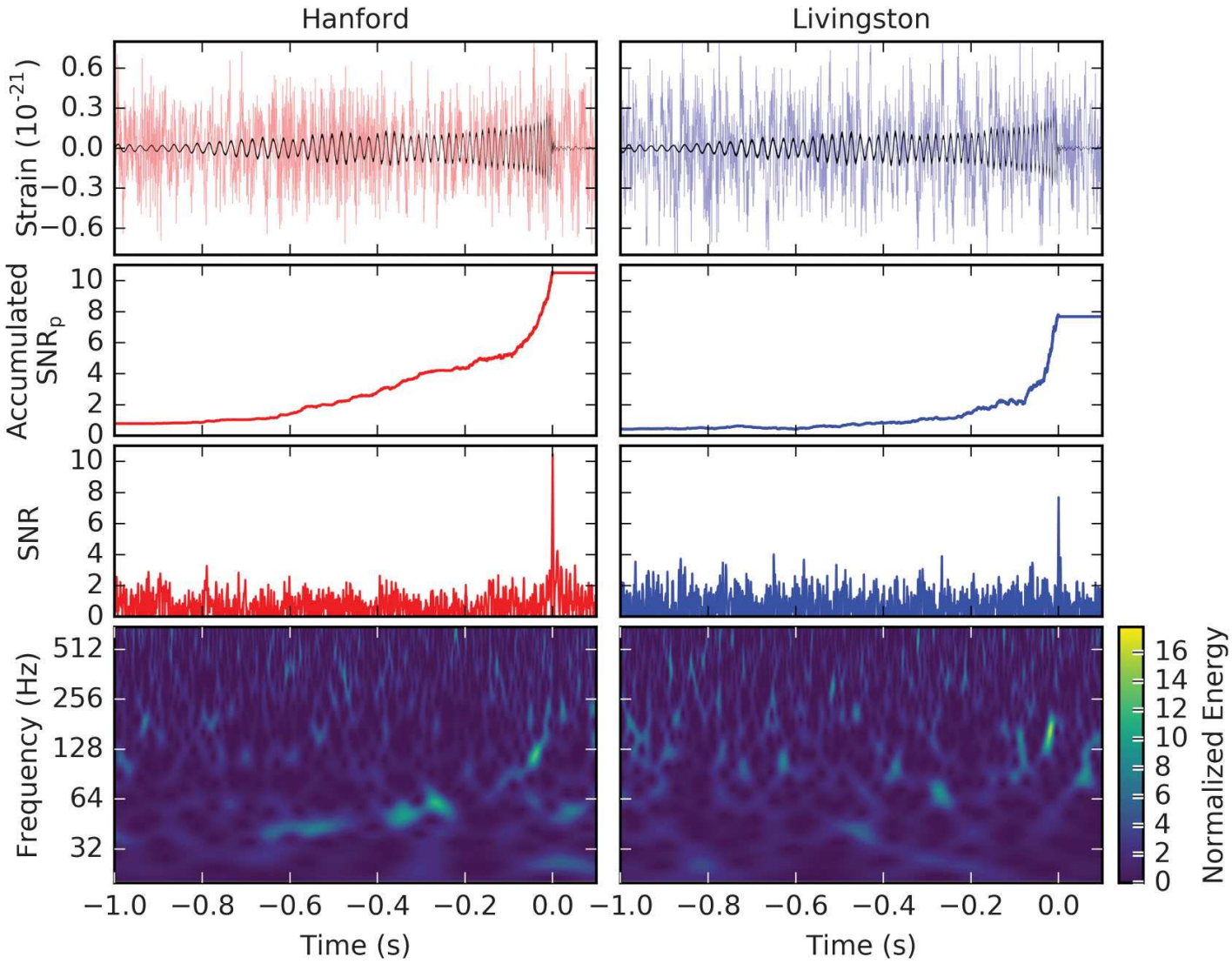
Need for accurate template waveforms





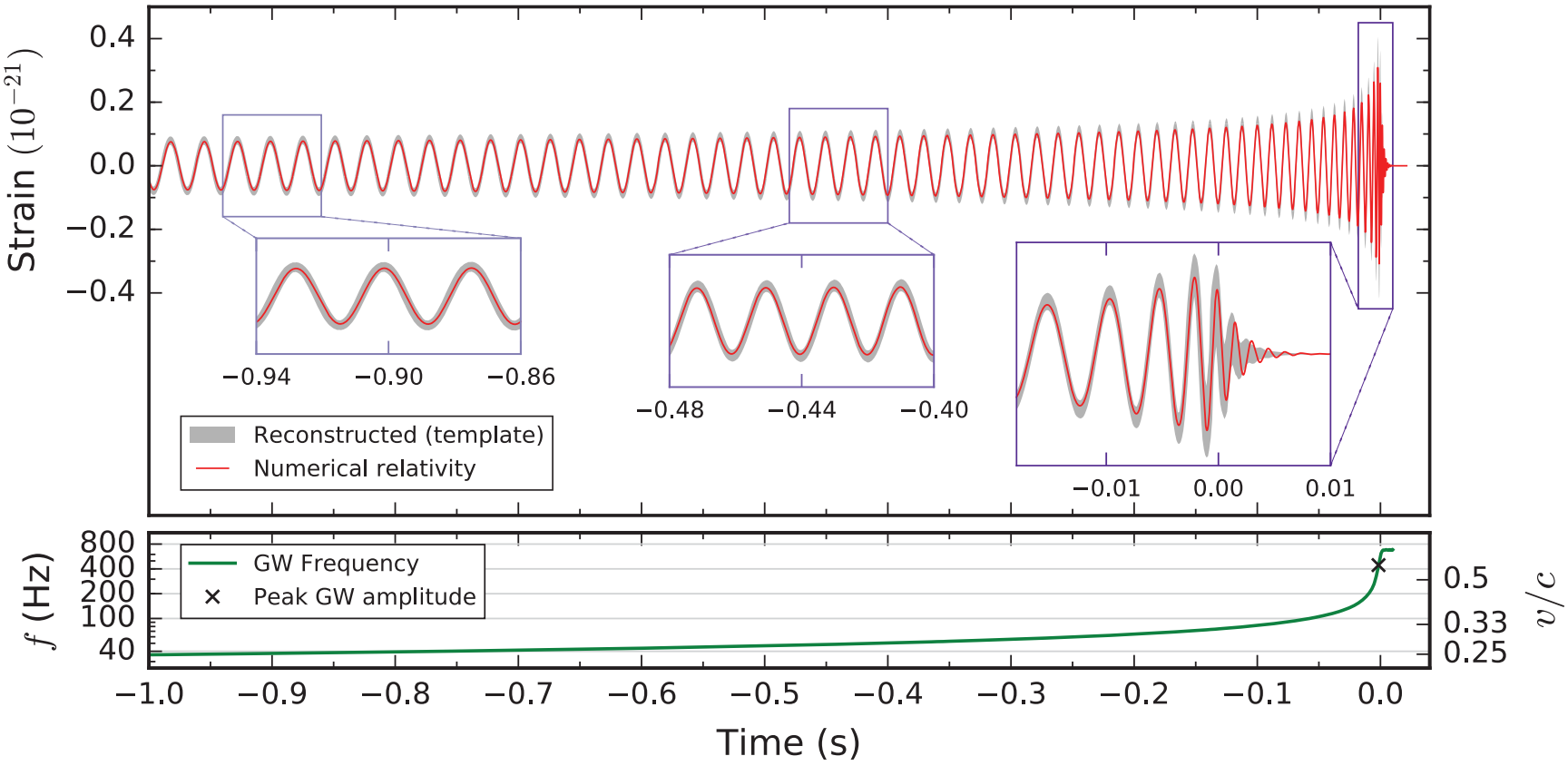
If the expected signal is *known in advance* then $n(t)$ can be filtered and $h(t)$ recovered by matched filtering $\longrightarrow$ **template waveforms**

A recent example: the event GW151226



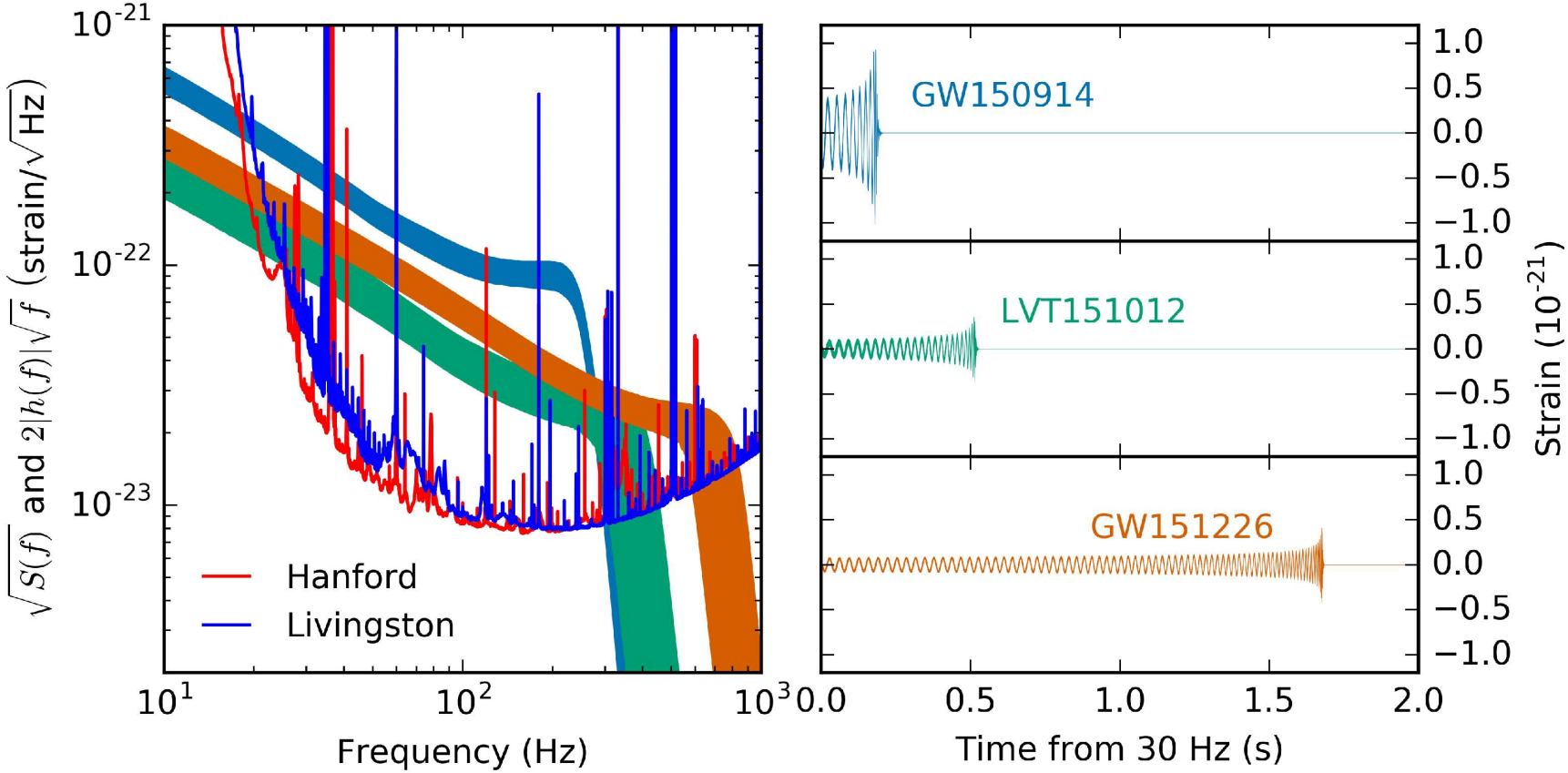
[PRL **116** (2016) 241103]

A long inspiral to merger to ringdown



[PRL **116** (2016) 241103]

The first three detections



[PRX **6** (2016) 041015]

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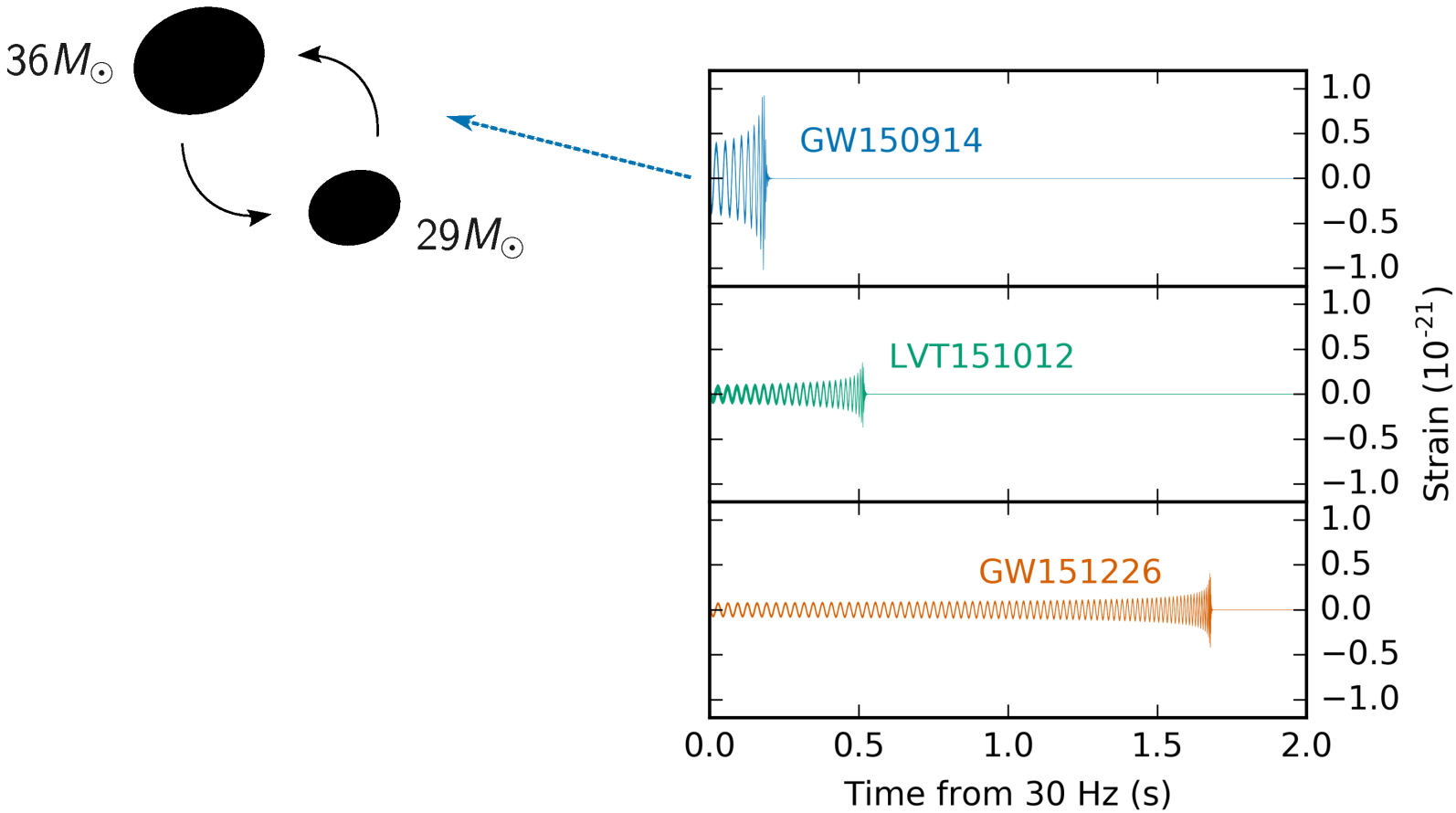

[PRX **6** (2016) 041015]

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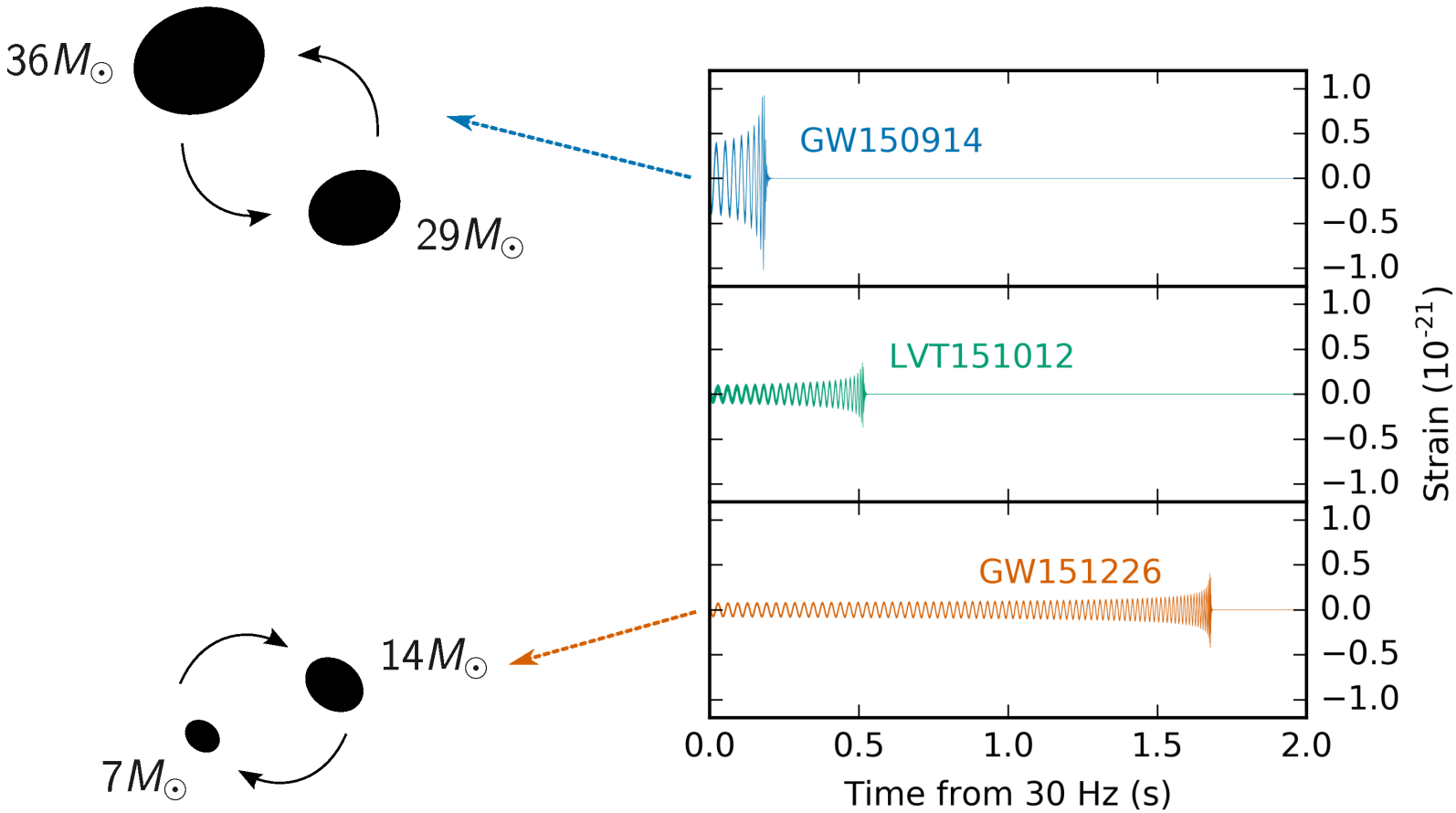

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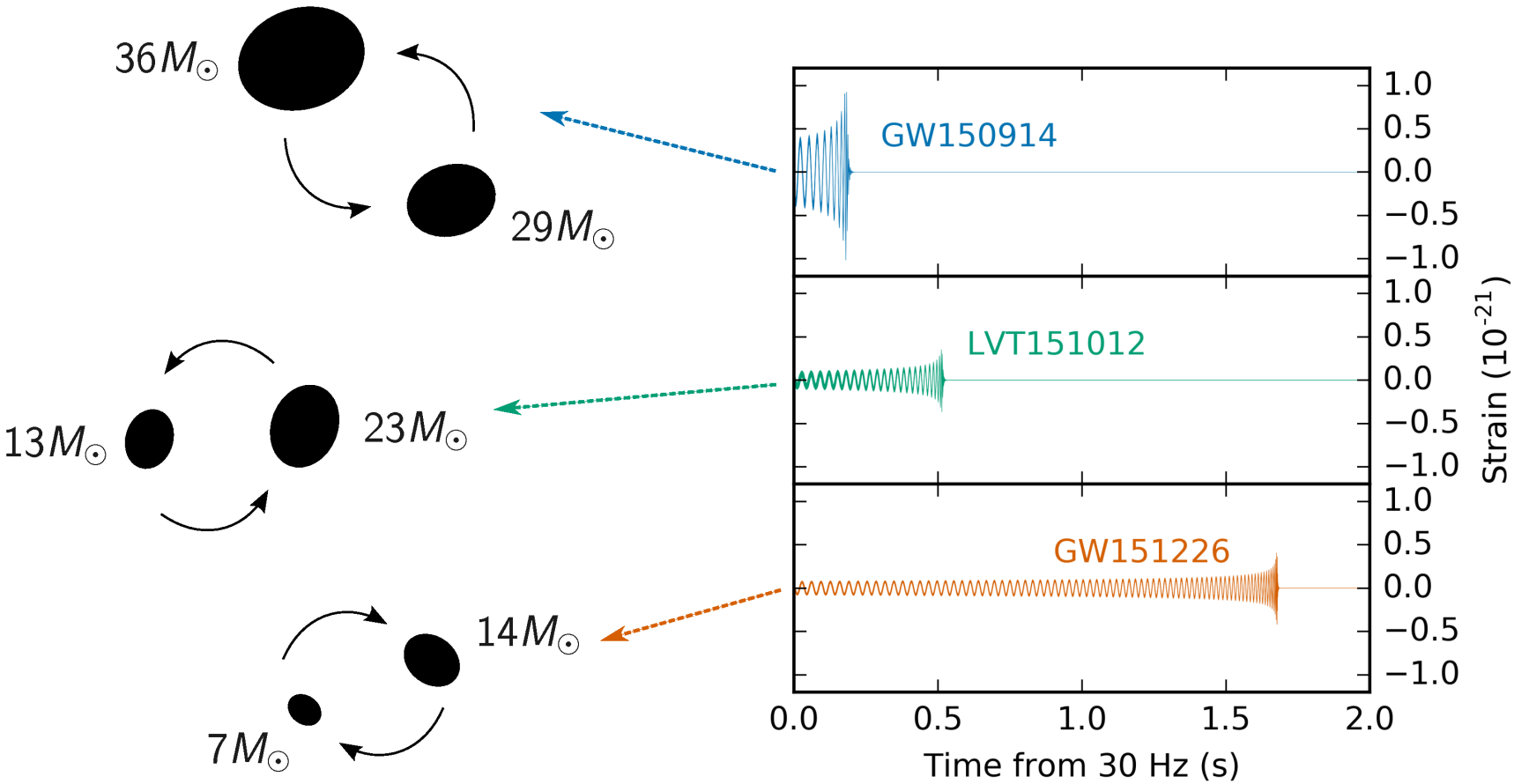


[PRX **6** (2016) 041015]

Outline

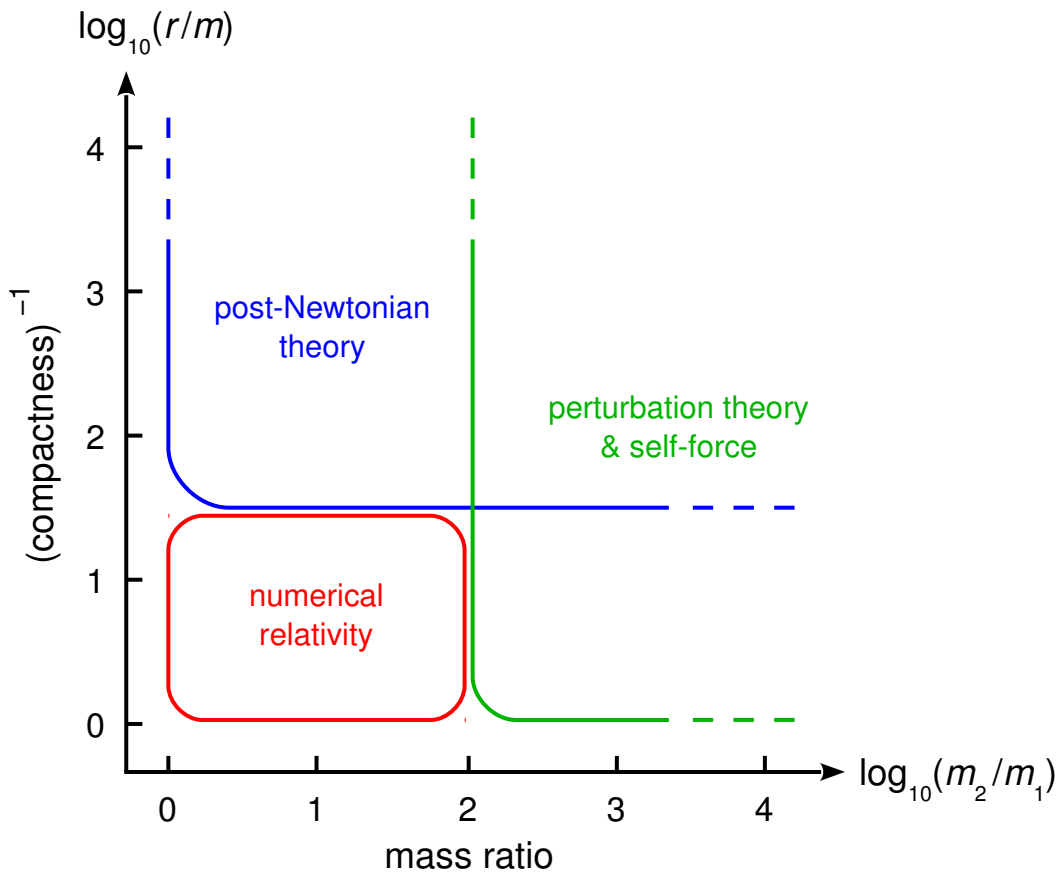
1 Context

2 Source modelling

3 EMRIs

4 Comparisons

Modelling coalescing compact binaries



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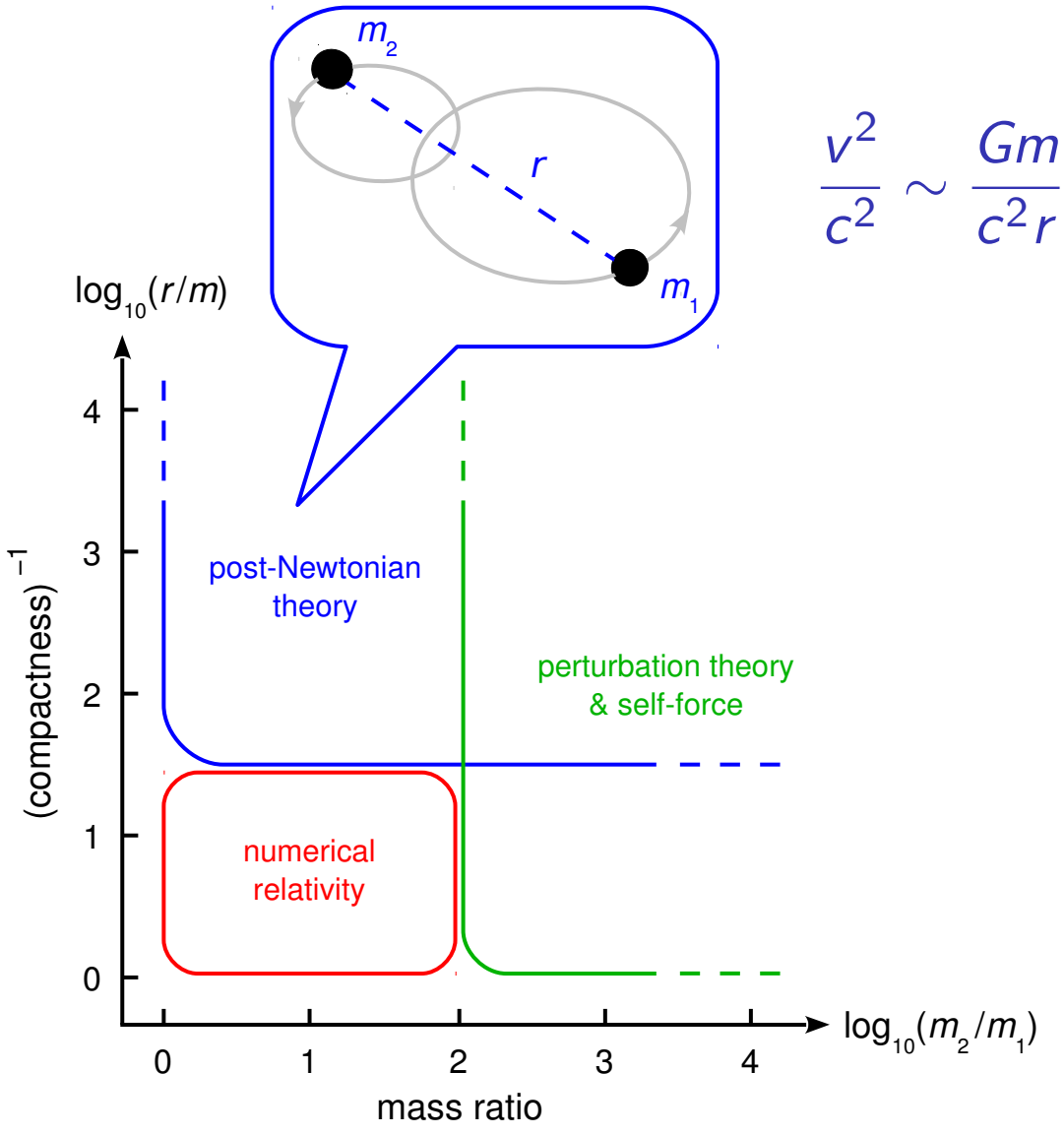
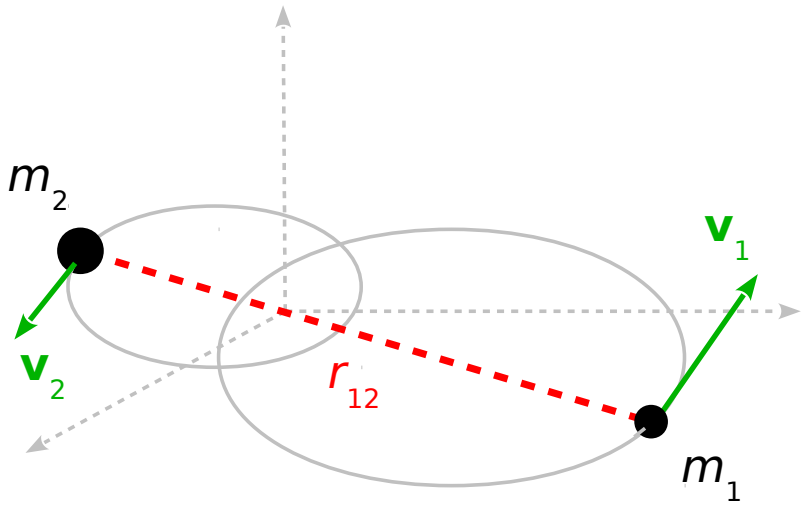


$$\frac{v^2}{c^2} \sim \frac{Gm}{c^2 r} \ll 1$$

Small parameter

$$\varepsilon \sim \frac{\mathbf{v}_{12}^2}{c^2} \sim \frac{Gm}{r_{12}c^2} \ll 1$$



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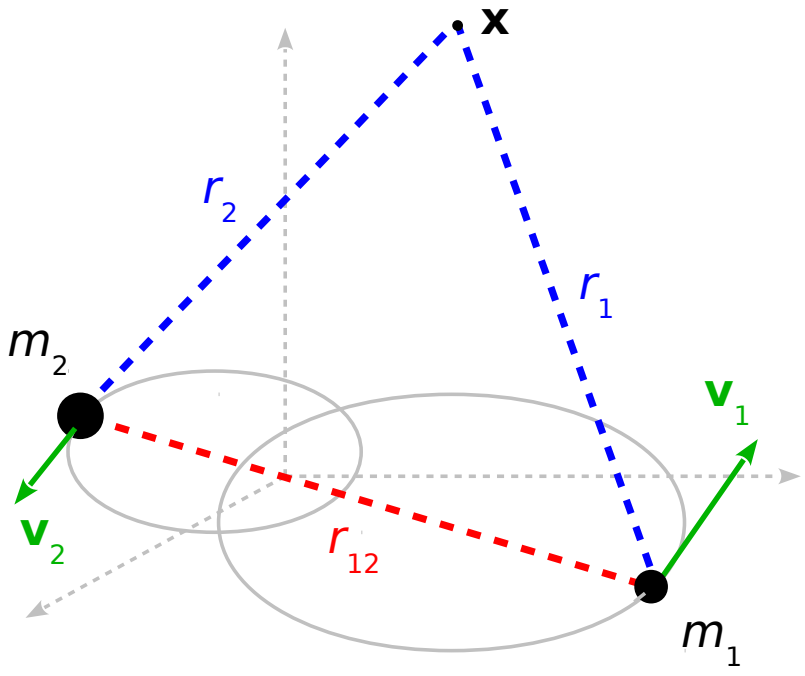

Example

$$g_{00}(t,\mathbf{x}) = -1 + \underbrace{\frac{2Gm_1}{r_1c^2}}_{\text{Newtonian}} + \underbrace{\frac{4Gm_2\mathbf{v}_2^2}{r_2c^4}}_{\text{1PN term}} + \cdots + (1 \leftrightarrow 2)$$

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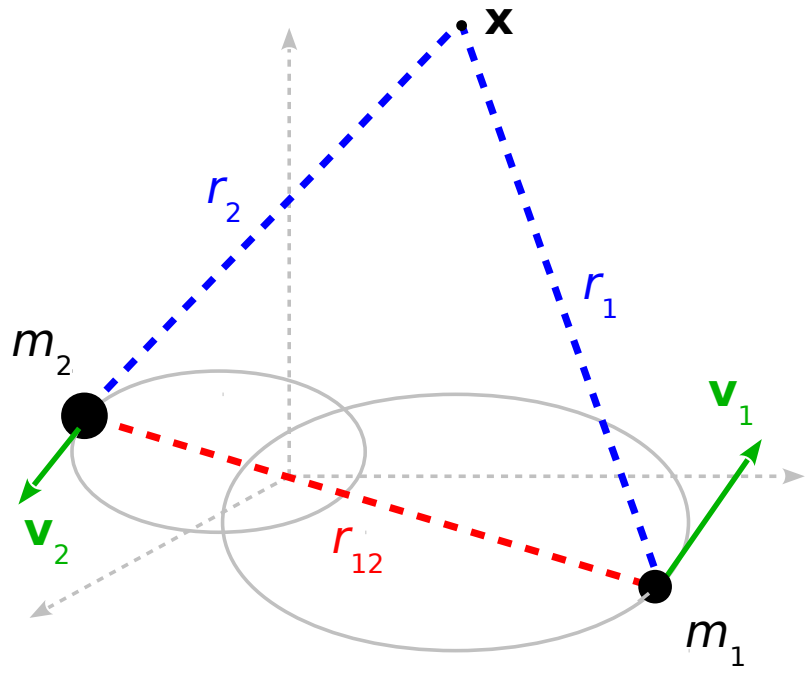

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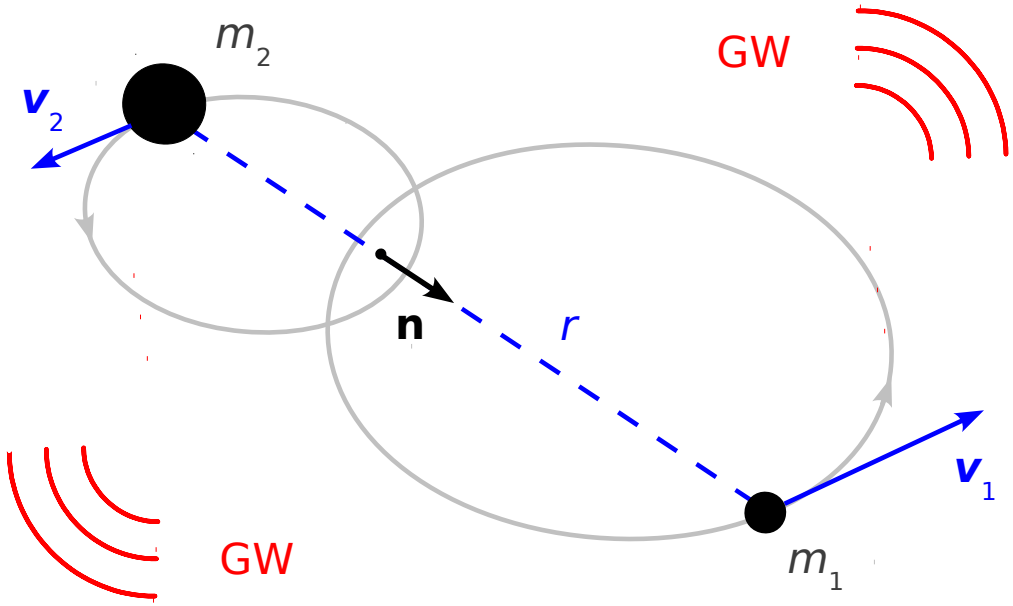
Notation

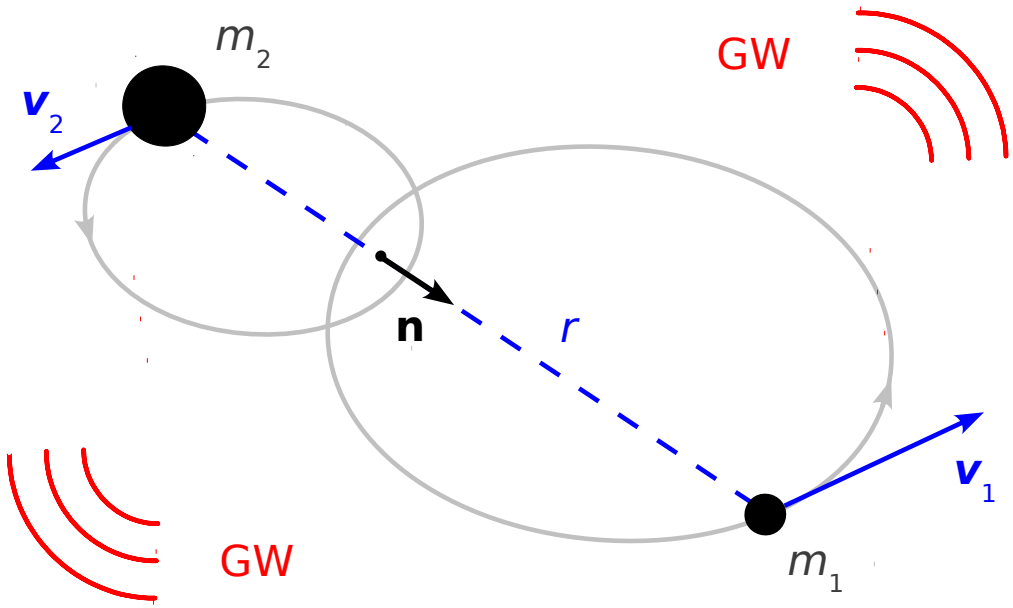
nPN order refers to effects $\mathcal{O}(c^{-2n})$ with respect to "Newtonian" solution

Post-Newtonian equations of motion



$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = \underbrace{-\frac{Gm_2}{r^2}\mathbf{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4}}_{\text{conservative terms}} + \underbrace{\frac{\boldsymbol{A}_{2.5\mathrm{PN}}}{c^5}}_{\text{rad. reac.}} + \underbrace{\frac{\boldsymbol{A}_{3\mathrm{PN}}}{c^6}}_{\text{cons. term}} + \underbrace{\frac{\boldsymbol{A}_{3.5\mathrm{PN}}}{c^7}}_{\text{rad. reac.}} + \cdots$$

Post-Newtonian equations of motion



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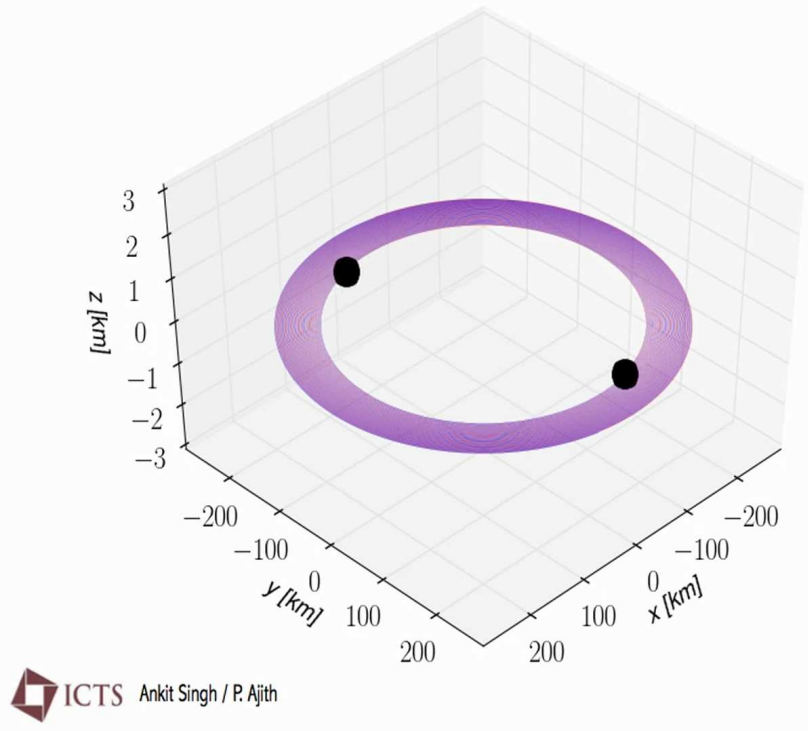
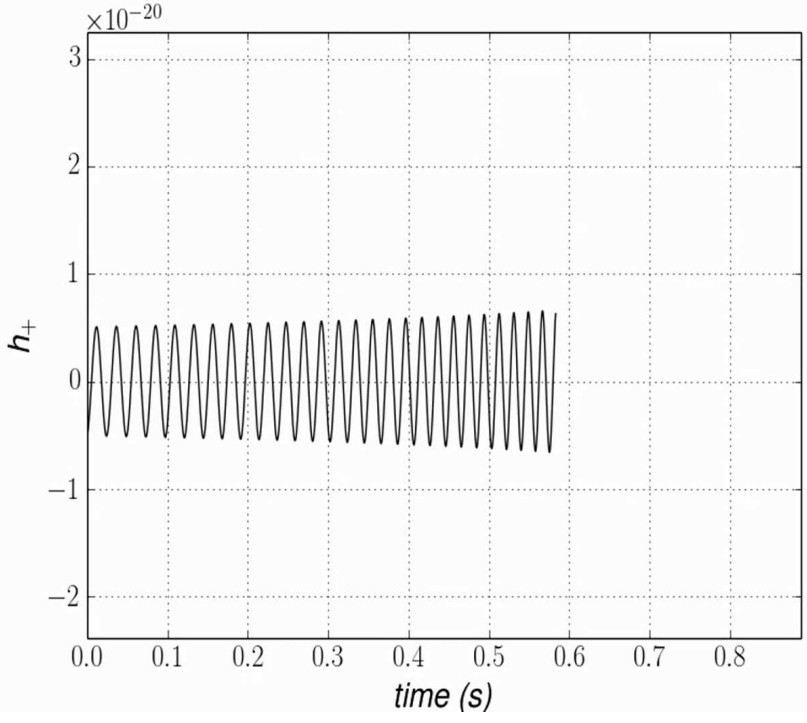
[see Luc Blanchet's talk]

Waveform for inspiralling compact binaries

Equal masses

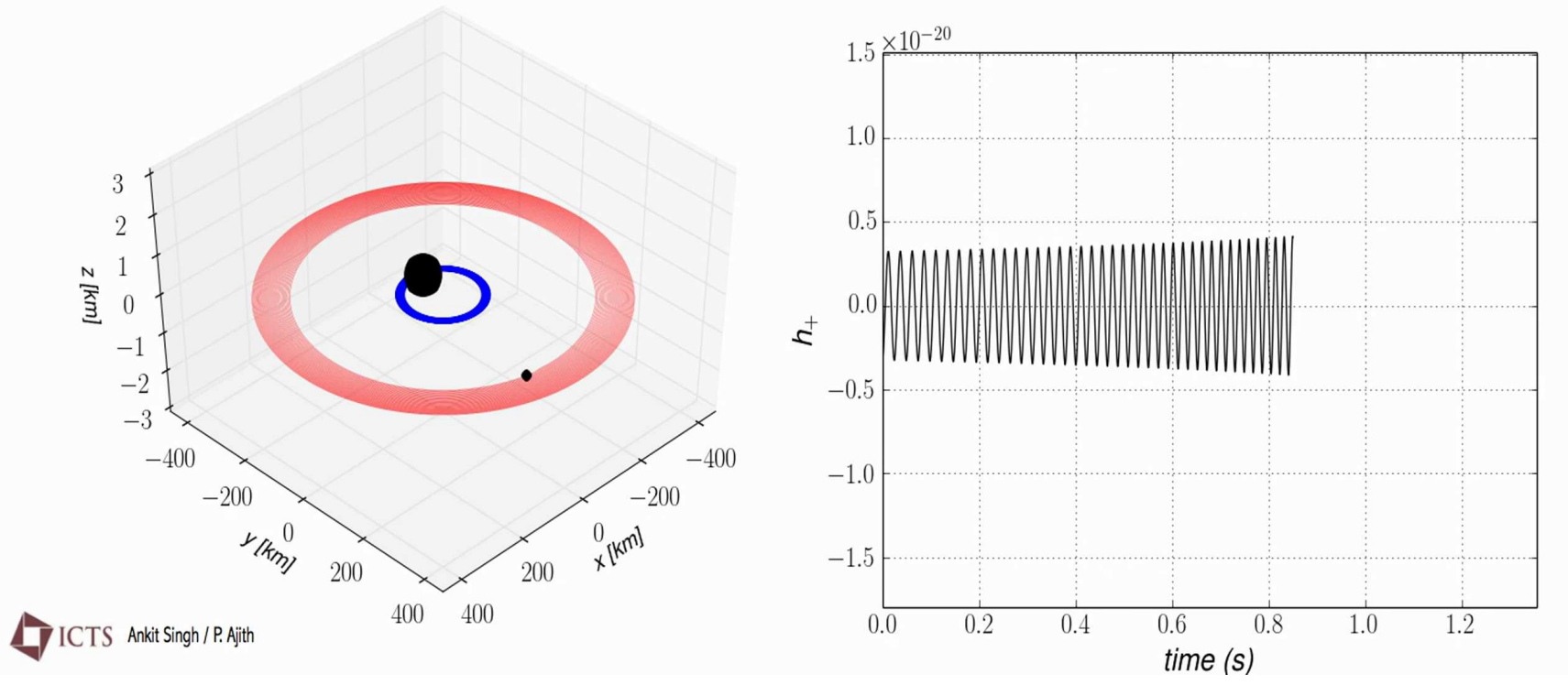




ICTS Ankit Singh / P. Ajith

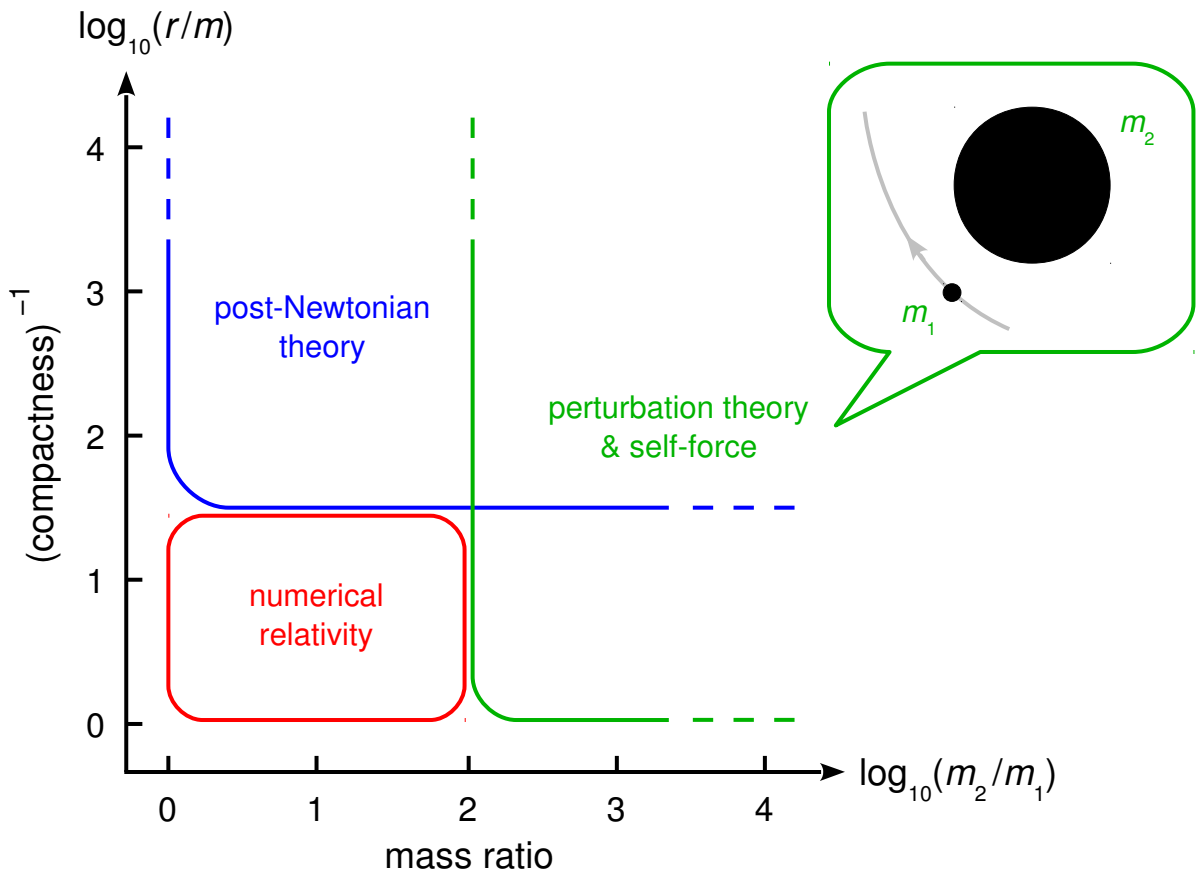
Waveform for inspiralling compact binaries

Unequal masses

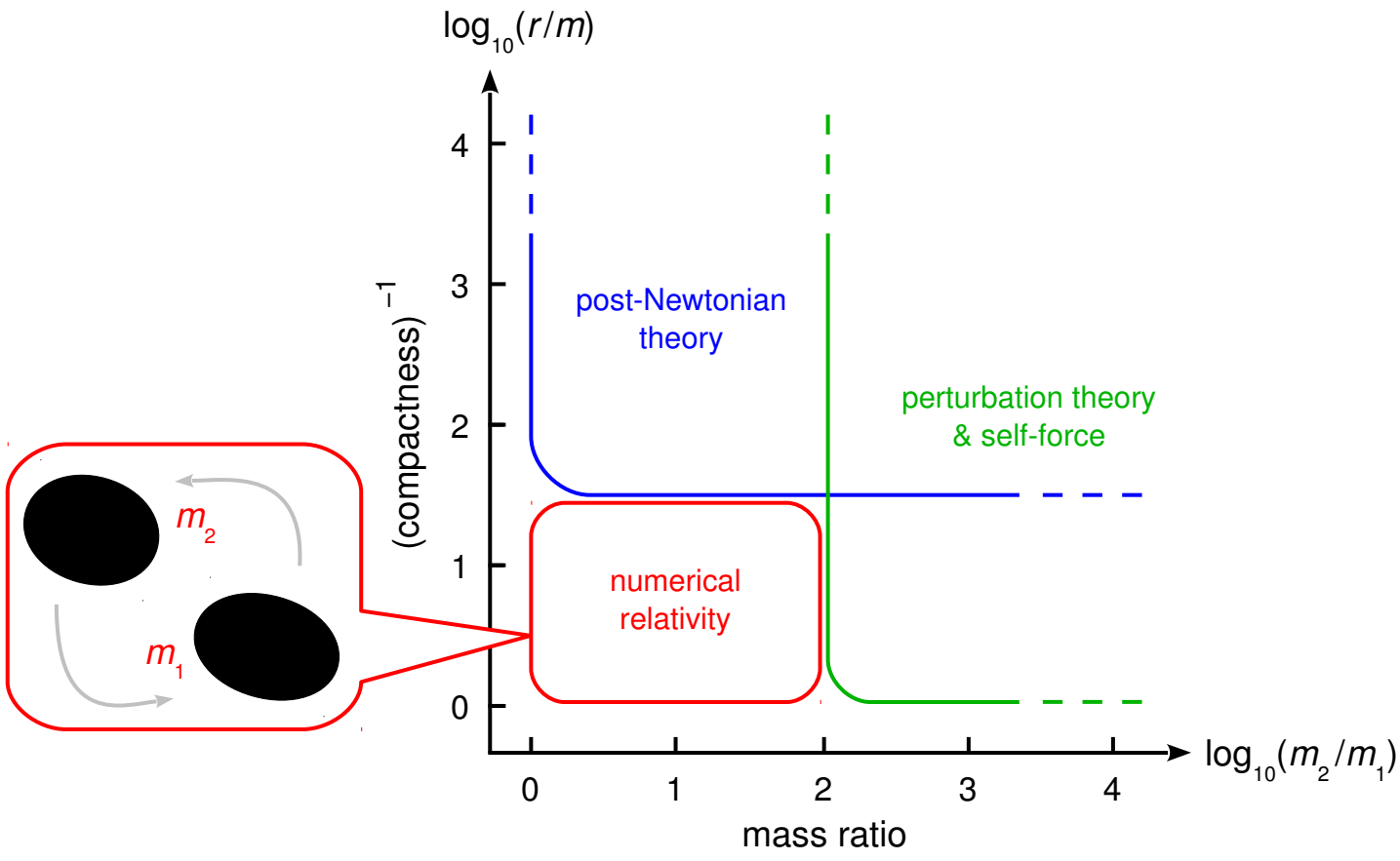


Modelling coalescing compact binaries

$$q \equiv \frac{m_1}{m_2} \ll 1$$



Modelling coalescing compact binaries

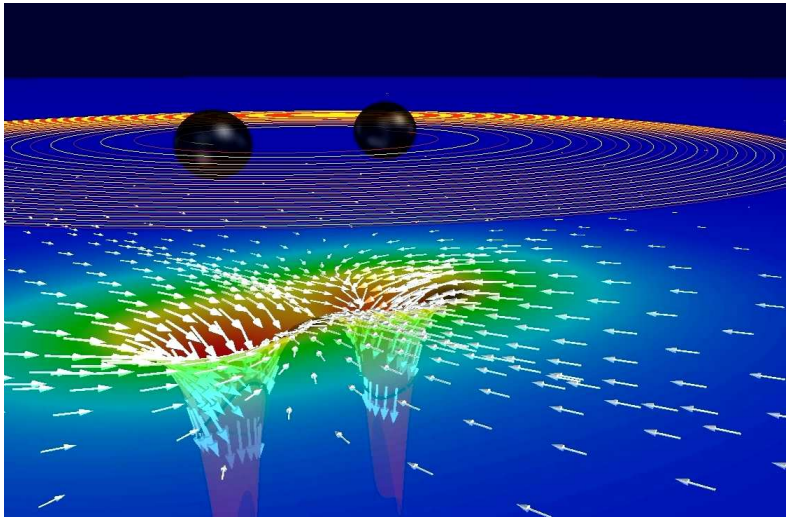


Numerical relativity simulations

- Numerical solution of the vacuum Einstein equations:

$$R_{\alpha\beta} = 0$$

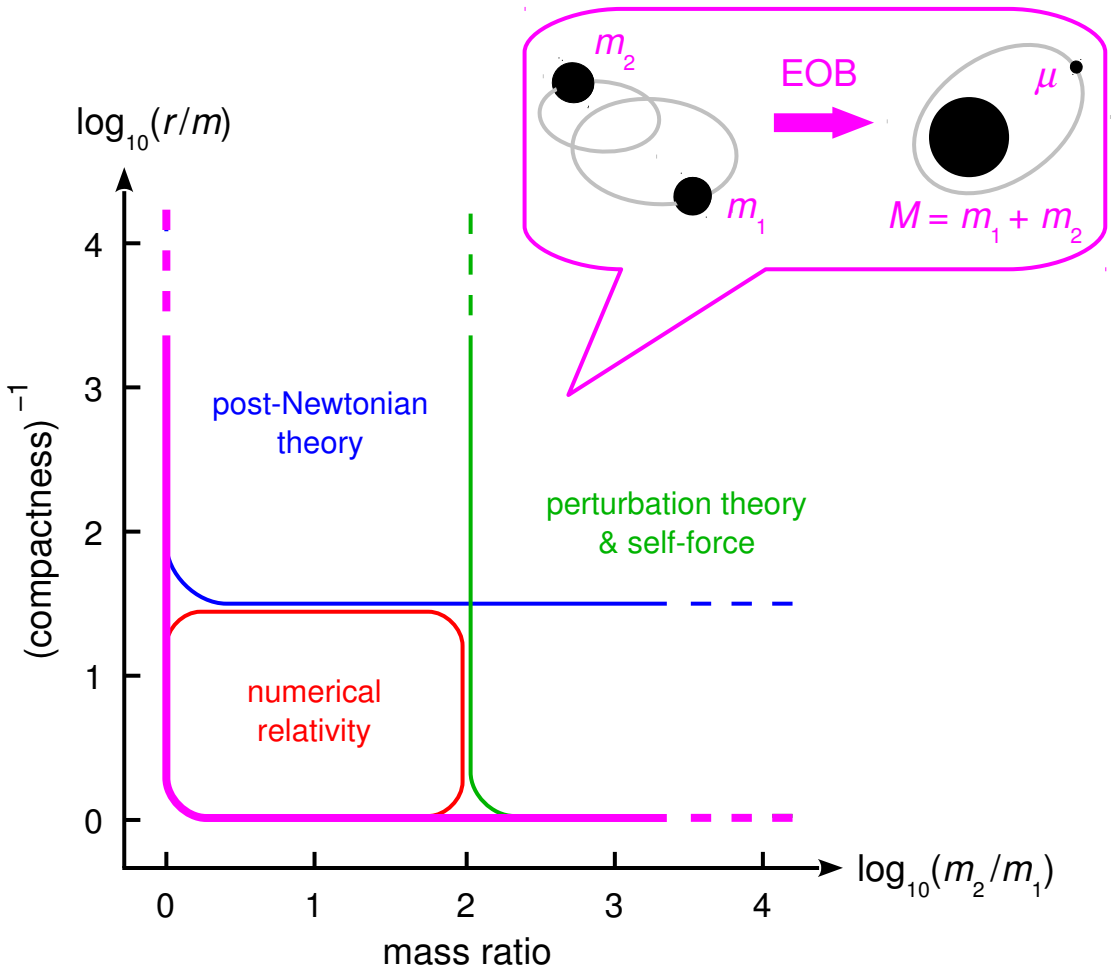
- Numerous challenges:
  - Initial data
  - Gauge freedom
  - Stable evolution
  - Large dynamic range
  - Black hole singularities
  - ...

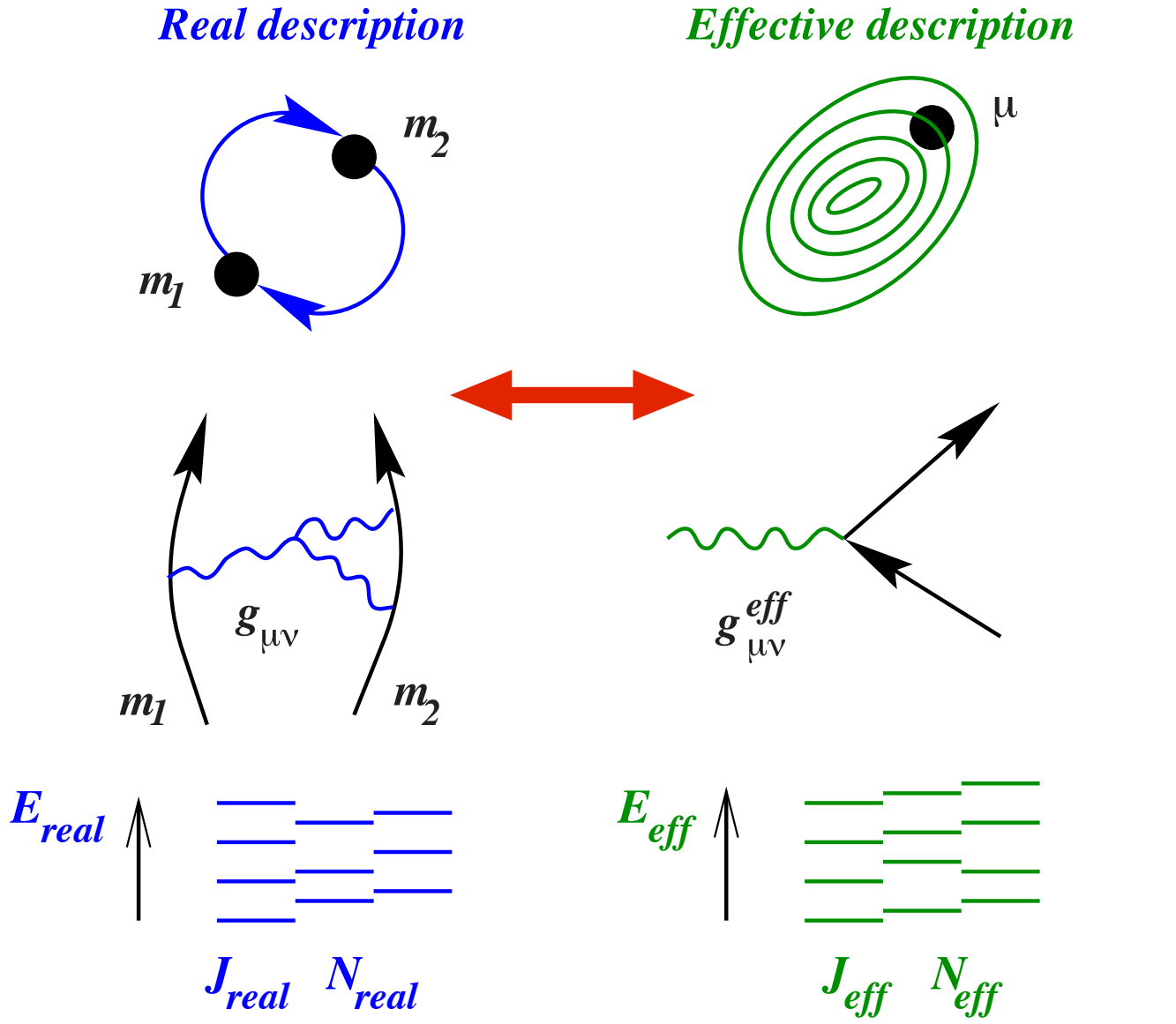


[SXS collaboration]

- Breakthrough in 2005 and significant progress ever since then

Modelling coalescing compact binaries





(Credit: Buonanno & Sathyaprakash 2015)

$$E_{\text{eff}}(J, N) = f(E_{\text{real}}(J, N))$$

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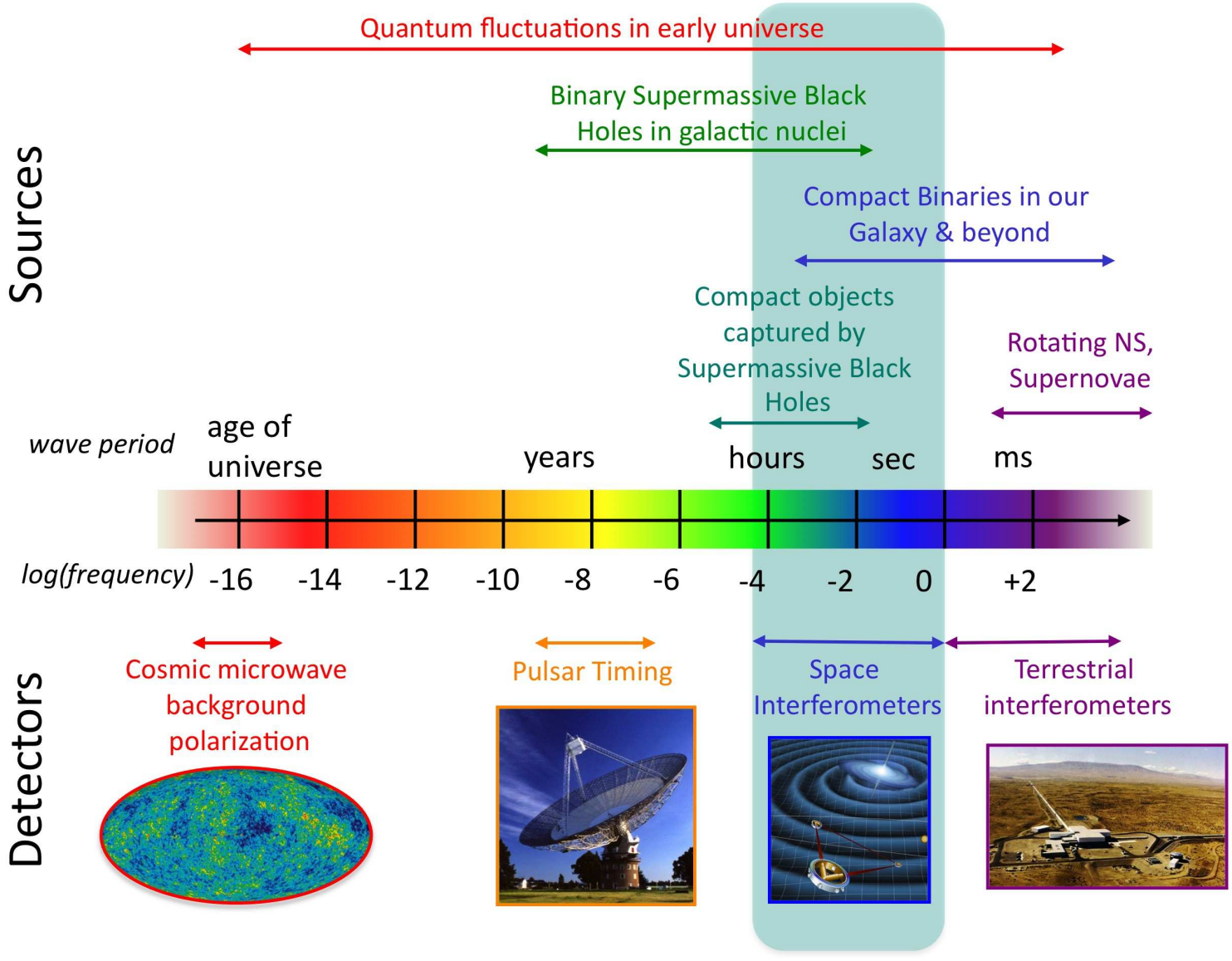
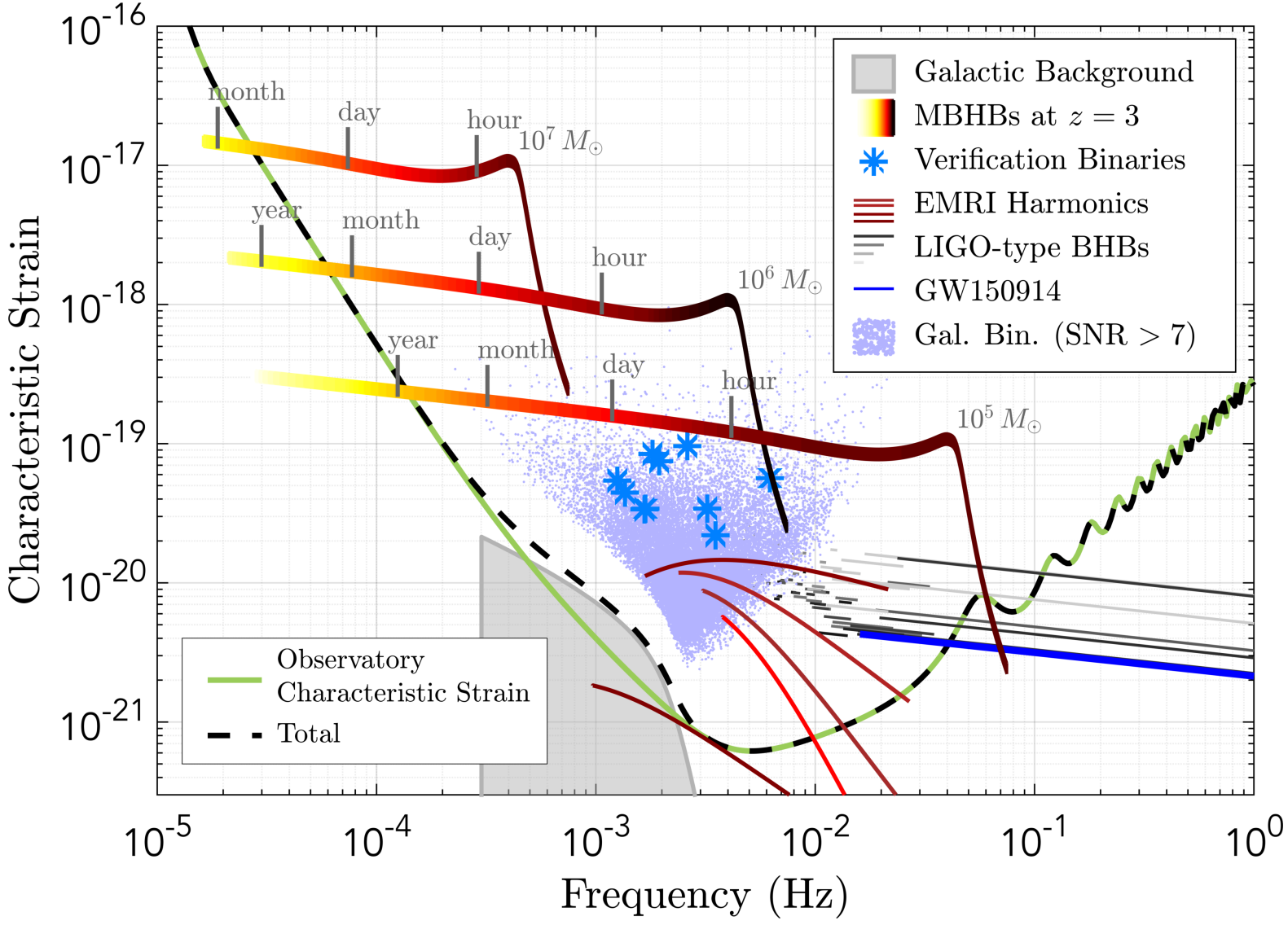
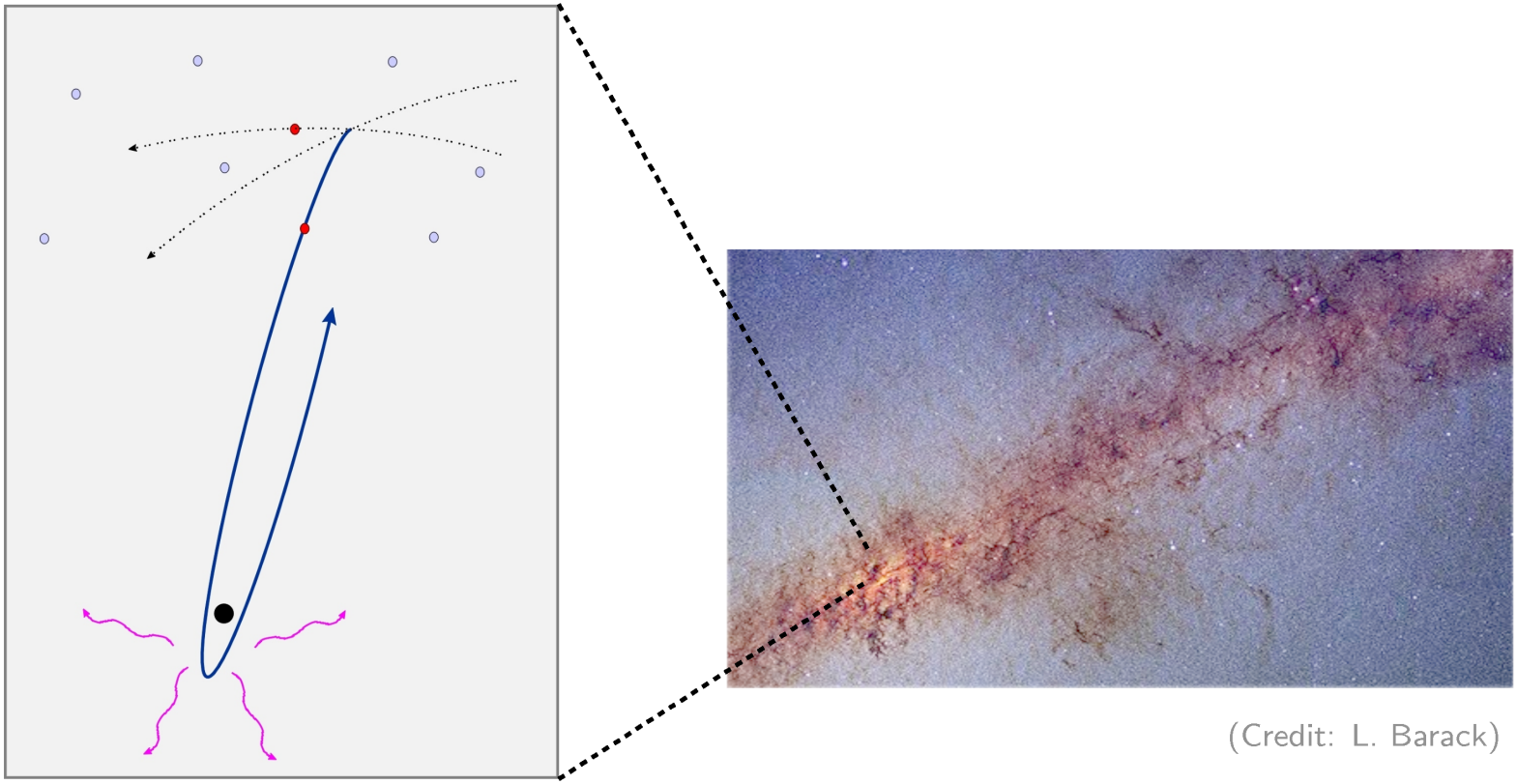
The gravitational-wave spectrum



LISA sources



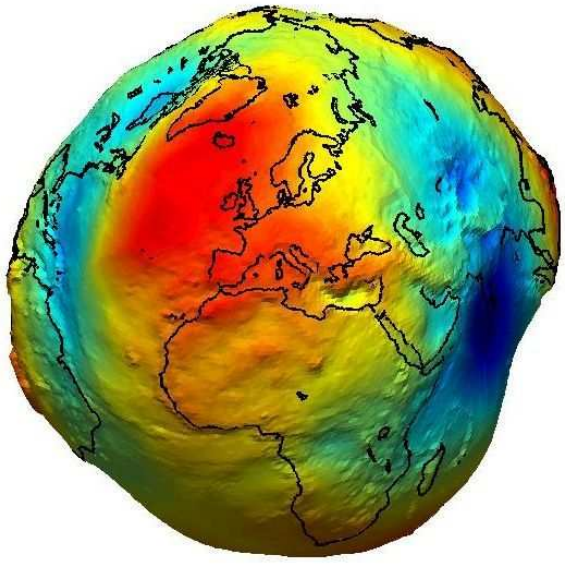
Extreme mass ratio inspirals



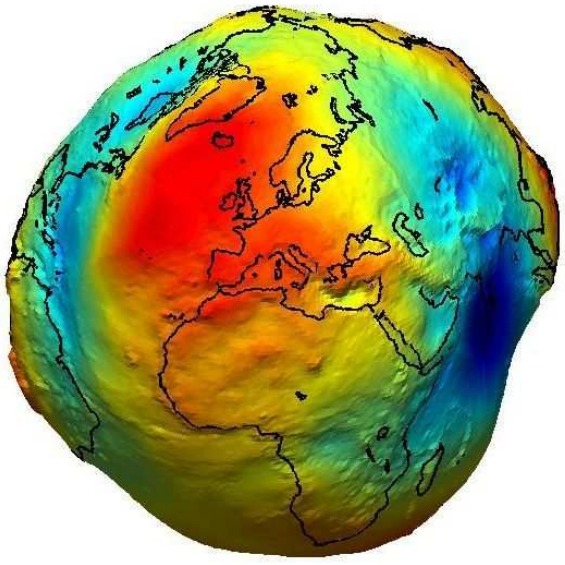
(Credit: L. Barack)

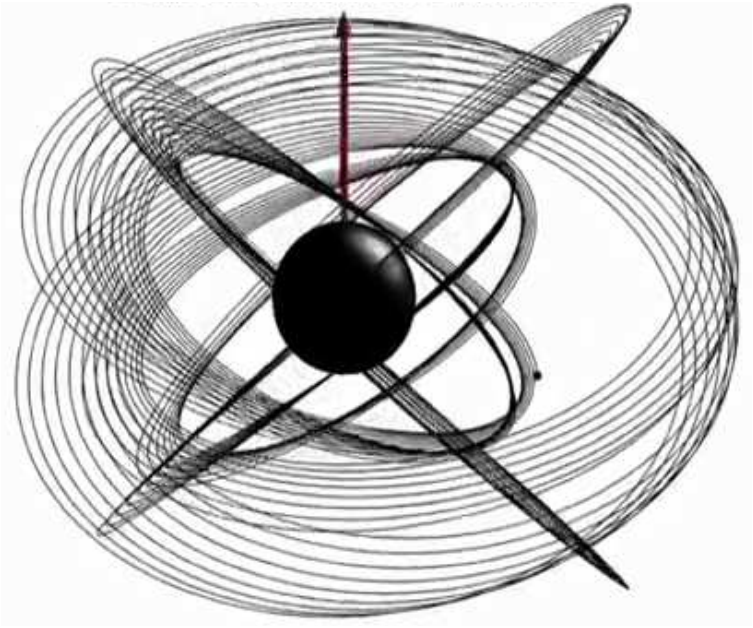
- LISA sensitive to $M_{\rm BH} \sim 10^5 - 10^7 M_\odot \rightarrow q \sim 10^{-7} - 10^{-4}$
- $T_{\rm orb} \propto M_{\rm BH} \sim$ hr and $T_{\rm insp} \propto M_{\rm BH}/q \sim$ yrs

Geodesy



Geodesy





(Credit: S. Drasco)

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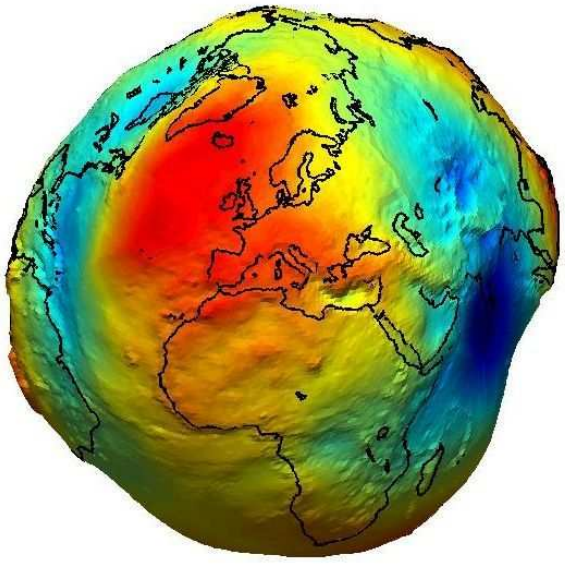
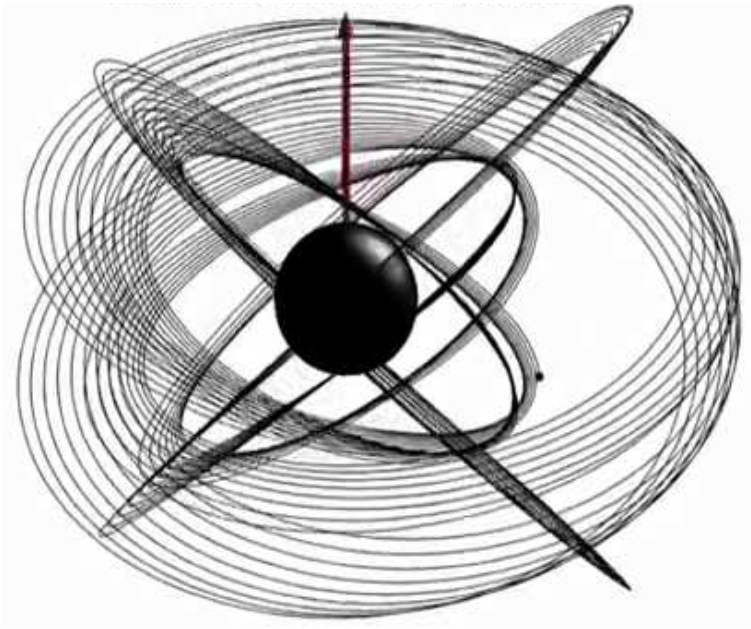


Botriomeladesy



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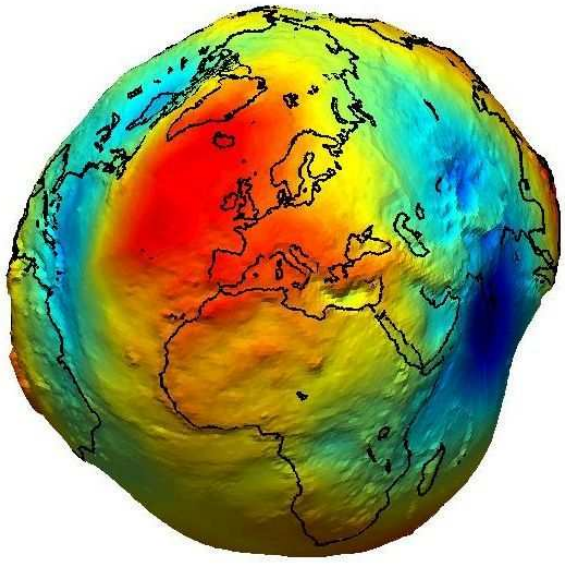

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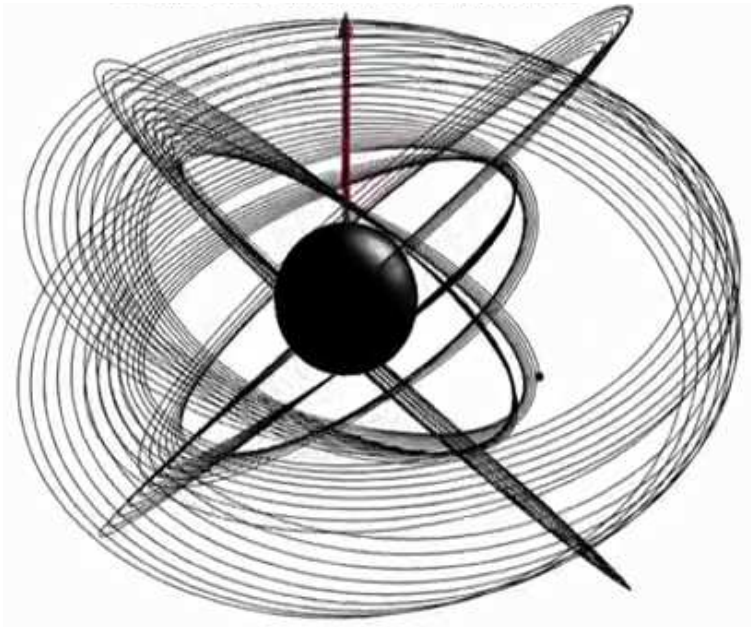
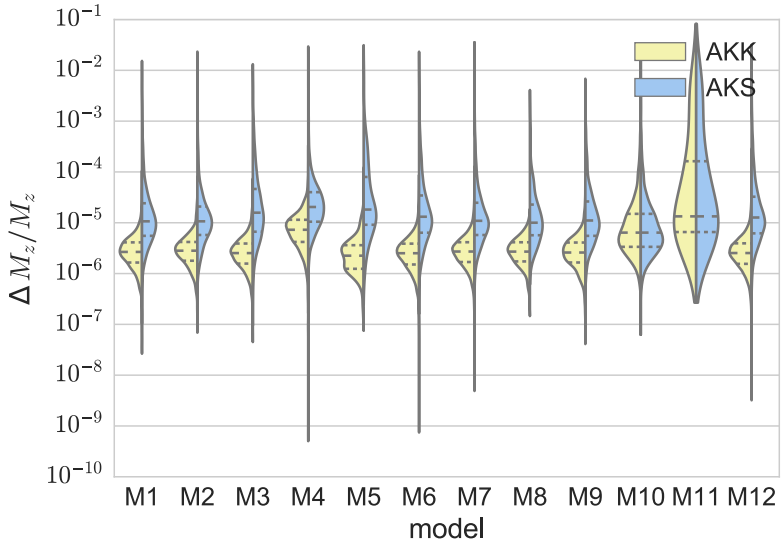
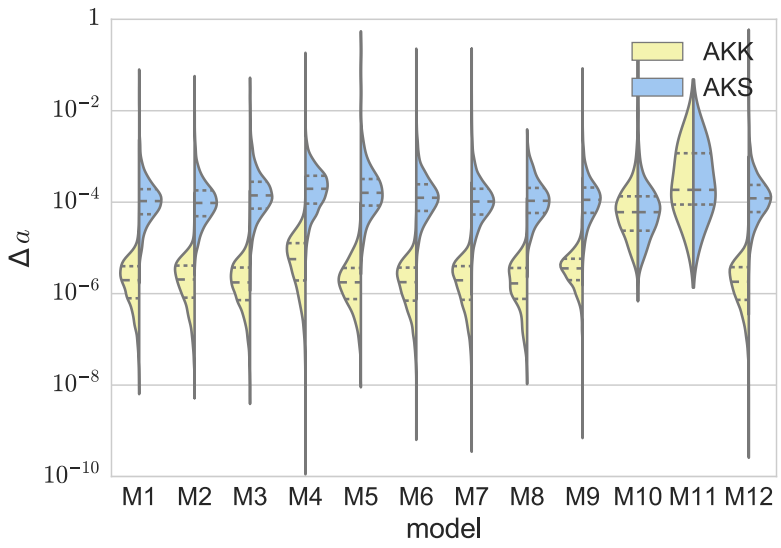


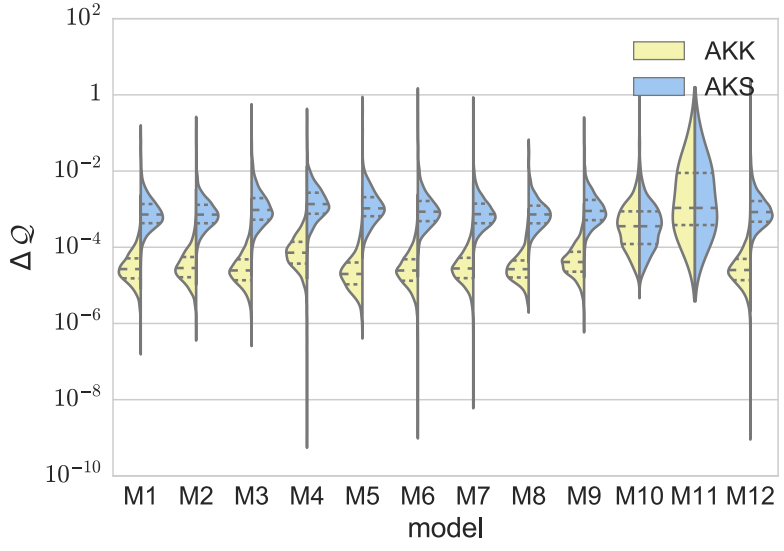
(Credit: S. Drasco)

Test of the black hole no-hair theorem

Testing the black hole no-hair theorem





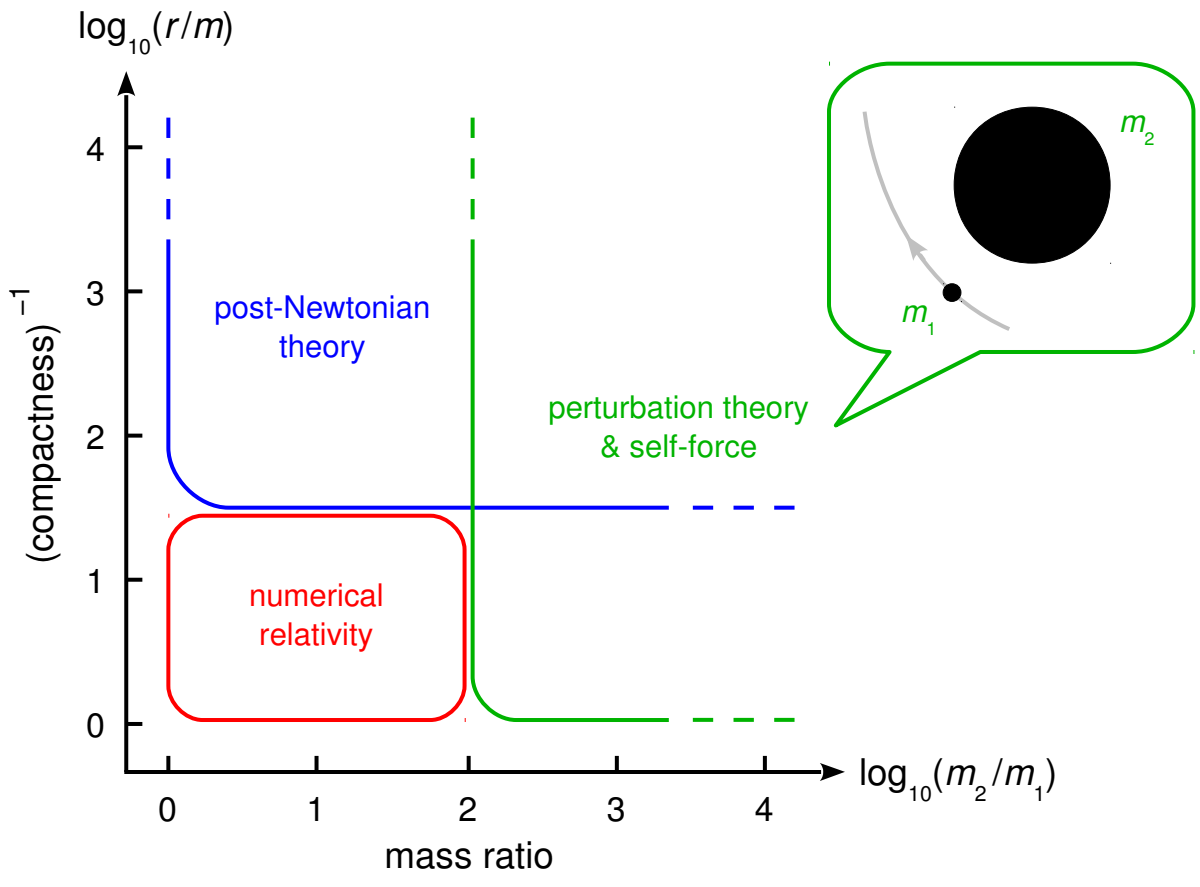


$$Q = -S^2/M$$

[PRD **95** (2017) 103012]

Large to extreme mass ratios

$$q \equiv \frac{m_1}{m_2} \ll 1$$



Black hole perturbation theory

Metric perturbation

$$g_{\alpha\beta} \equiv \underbrace{\mathring{g}_{\alpha\beta}}_{\text{bkgd}} + \underbrace{h_{\alpha\beta}}_{\text{pert.}}$$

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Einstein field equations

$$\Box\bar{h}_{\alpha\beta} + \underbrace{2\mathring{R}^{\mu\ \ \nu}_{\ \alpha\ \ \beta}\bar{h}_{\mu\nu}}_{\substack{\text{coupling to} \\ \text{curvature}}} = -\underbrace{16\pi T_{\alpha\beta}}_{\text{source}}$$

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Kerr background

- Stationarity $\rightarrow$ Fourier mode decomposition $e^{-\mathrm{i}\omega t}$
- Axial symmetry $\rightarrow$ spin-weighted spheroidal harm. ${}_{-2}S^{a\omega}_{\ell m}(\theta)e^{\mathrm{i}m\phi}$
- Teukolsky equation for ψ_0/ψ_4 is **separable**

Black hole perturbation theory

Metric perturbation

$$g_{\alpha\beta} \equiv \underbrace{\mathring{g}_{\alpha\beta}}_{\text{bkgd}} + \underbrace{h_{\alpha\beta}}_{\text{pert.}}$$

Einstein field equations

$$\Box\bar{h}_{\alpha\beta} + \underbrace{2\mathring{R}^{\mu\ \ \nu}_{\ \alpha\ \ \beta}\bar{h}_{\mu\nu}}_{\substack{\text{coupling to}\\ \text{curvature}}} = -\underbrace{16\pi T_{\alpha\beta}}_{\text{source}}$$

Kerr background

- Stationarity $\rightarrow$ Fourier mode decomposition $e^{-\mathrm{i}\omega t}$
- Axial symmetry $\rightarrow$ spin-weighted spheroidal harm. ${}_{-2}S^{a\omega}_{\ell m}(\theta)e^{\mathrm{i}m\phi}$
- Teukolsky equation for ψ_0/ψ_4 is **separable**

Gravitational waveform + Fluxes

Geodesic motion in Kerr spacetime

Canonical Hamiltonian

$$H(x,p) = \frac{1}{2}\,\mathring{g}^{\alpha\beta}(x)\,p_\alpha p_\beta$$

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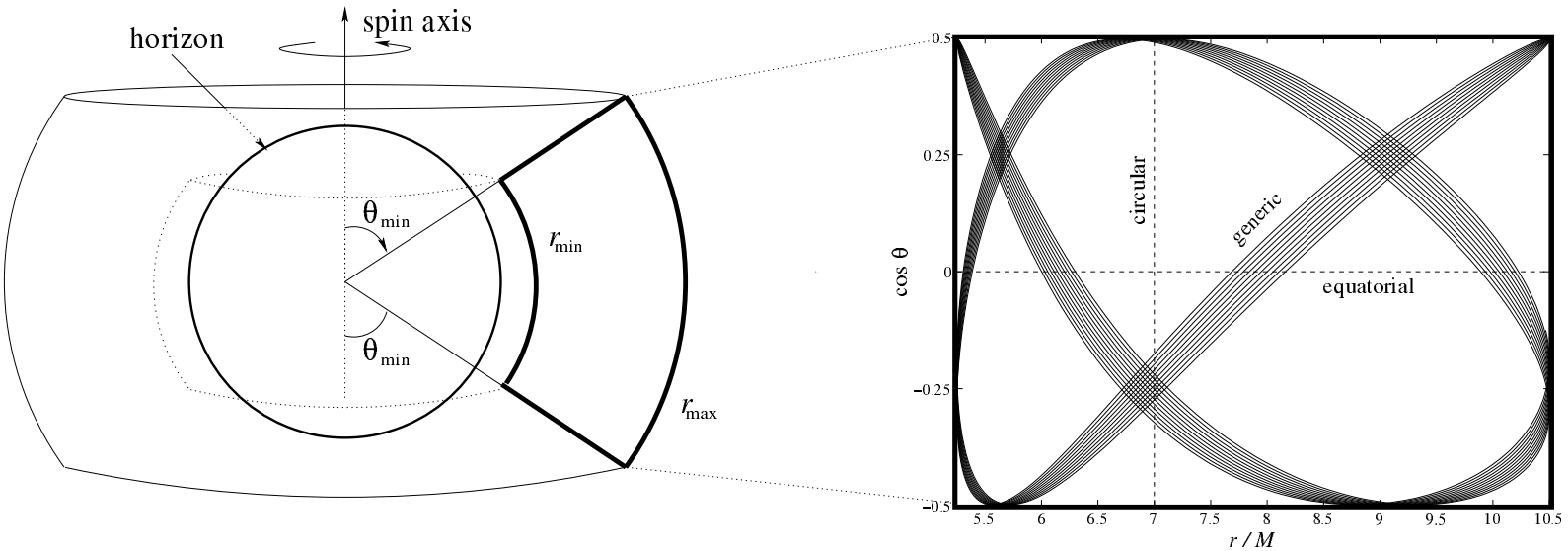
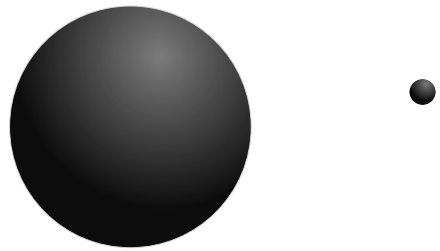


(Credit: Drasco & Hughes 2006)

Adiabatic approximation

$$\frac{T_{\mathrm{insp}}}{T_{\mathrm{orb}}} \propto \frac{1}{q} \gg 1$$



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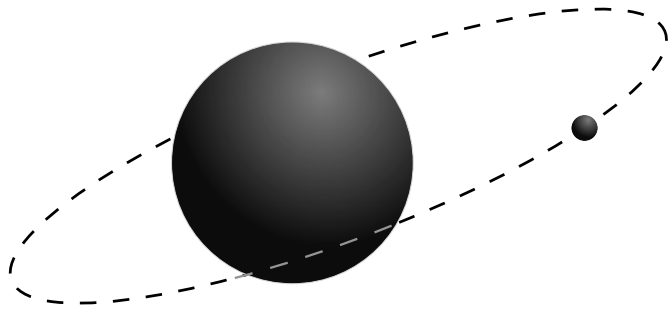

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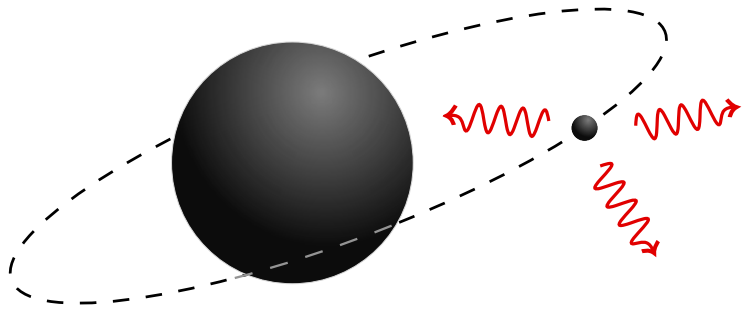

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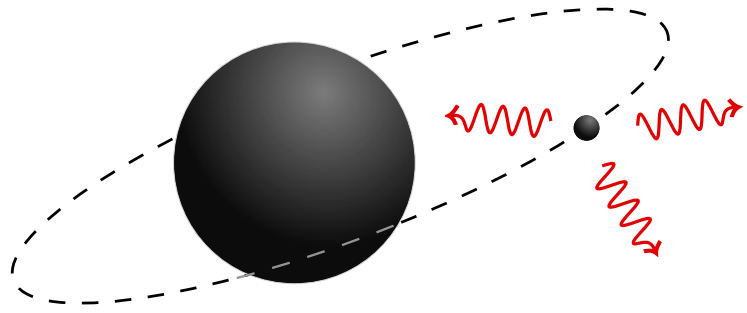

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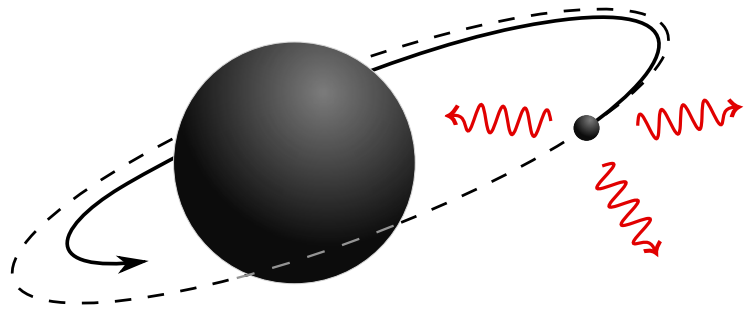

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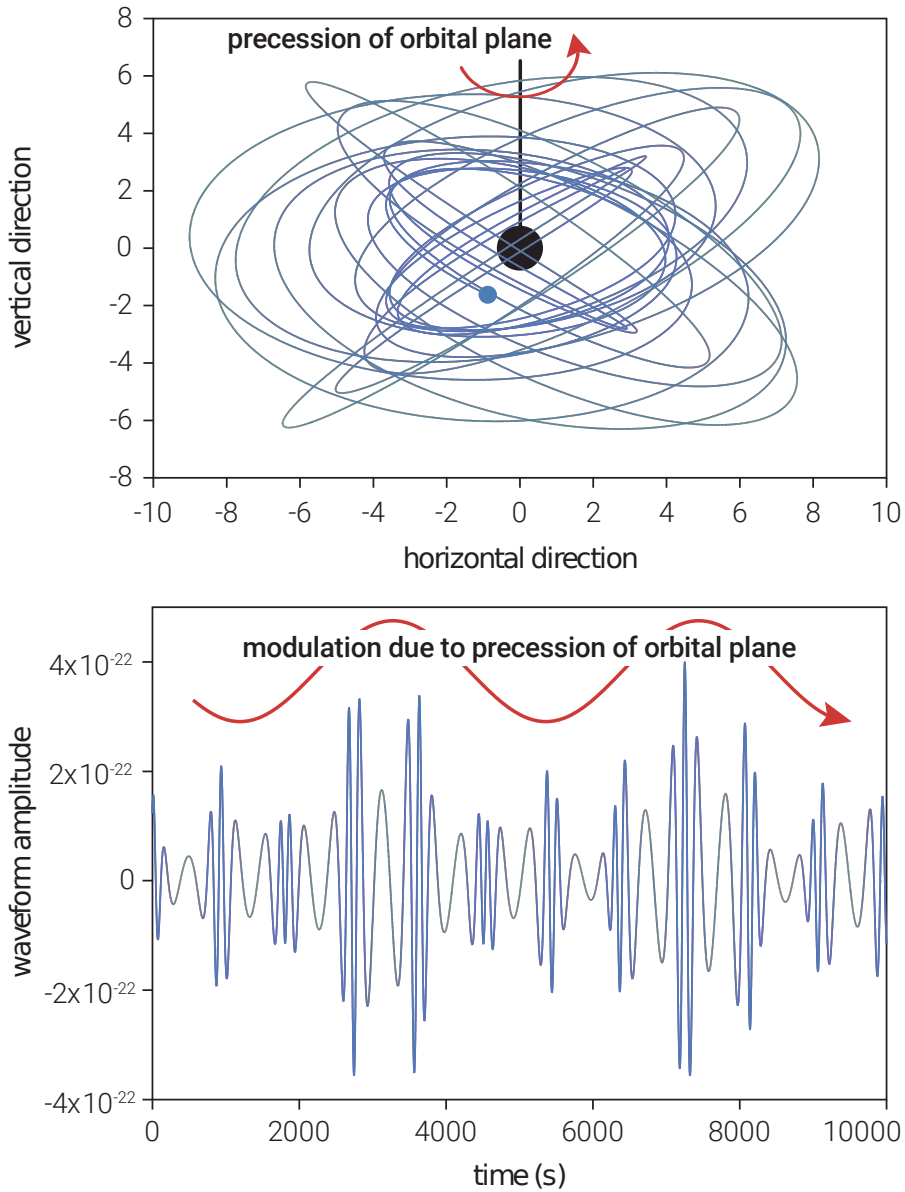
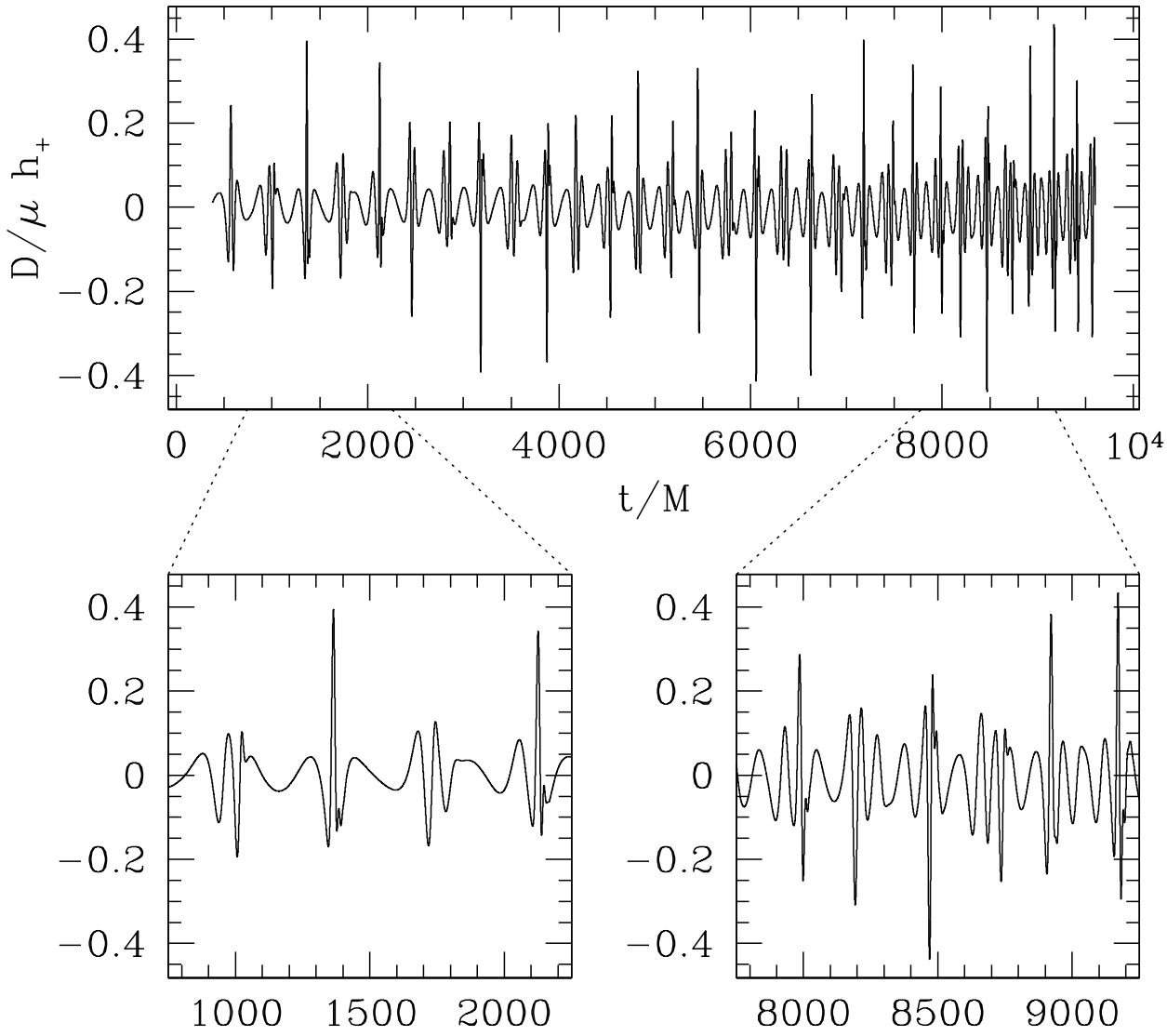
- **Update** the orbit and play again!

Waveform in the adiabatic approximation



[LISA whitepaper]

Waveform in the adiabatic approximation



[PRD **78** (2008) 024022]

Accuracy requirement for the phasing

- Over an inspiral timescale $T_{\rm insp} \sim M_{\rm BH}/q$, the GW phase is given by the expansion

$$\phi = \frac{1}{q}\left[\phi_0 + q\,\phi_1 + \mathcal{O}(q^2)\right]$$

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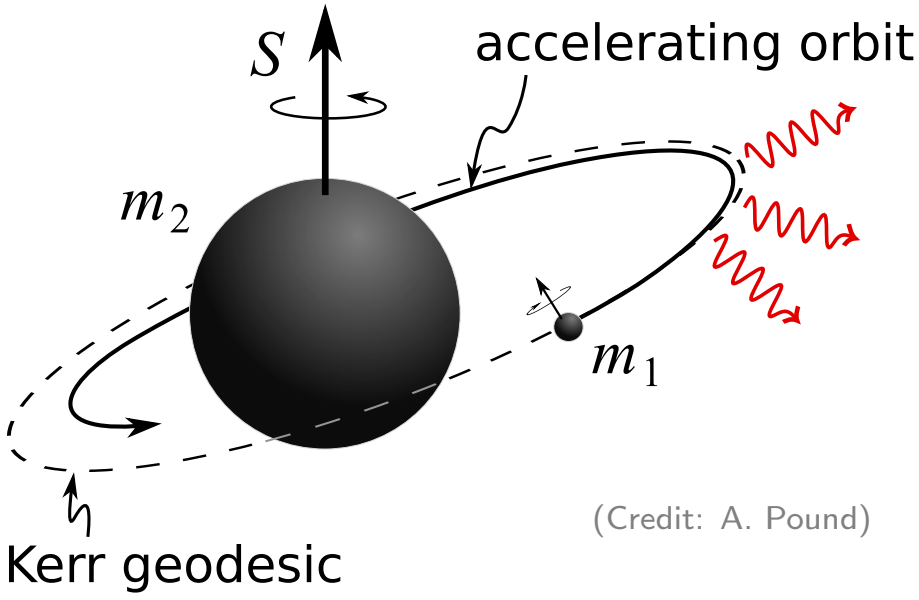
We need to account for the local effects of the metric perturbation on the body's orbital motion

Gravitational self-force



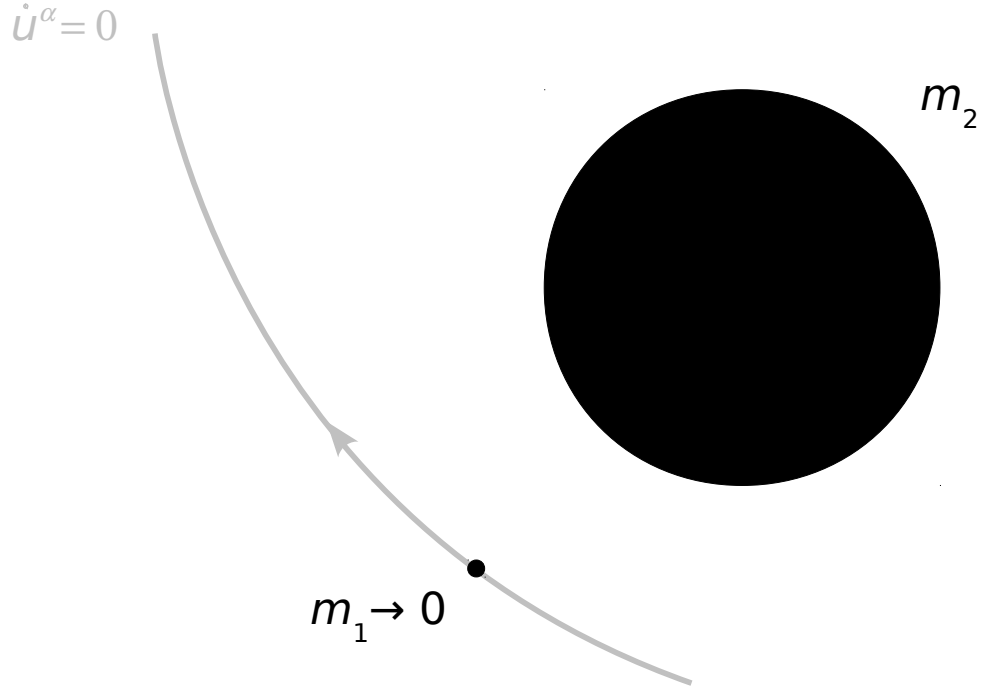
(Credit: A. Pound)

- Dissipative component $\longleftrightarrow$ gravitational waves
- Conservative component $\longleftrightarrow$ secular effects

Gravitational self-force

Spacetime metric

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta}$$



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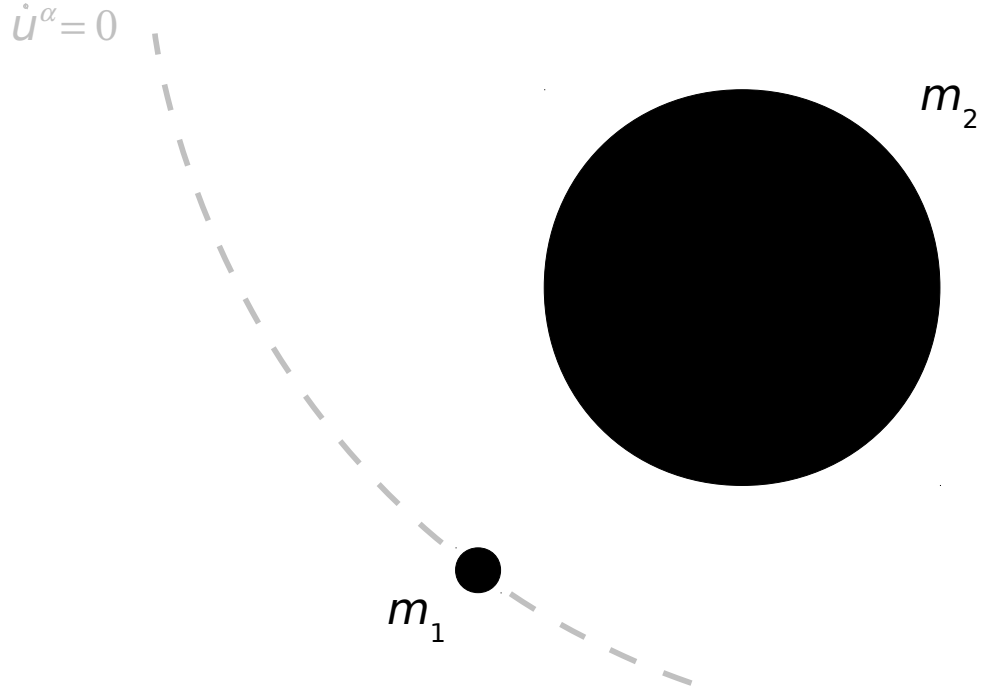
$$q \equiv \frac{m_1}{m_2} \ll 1$$



Gravitational self-force

Spacetime metric

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + h_{\alpha\beta}$$

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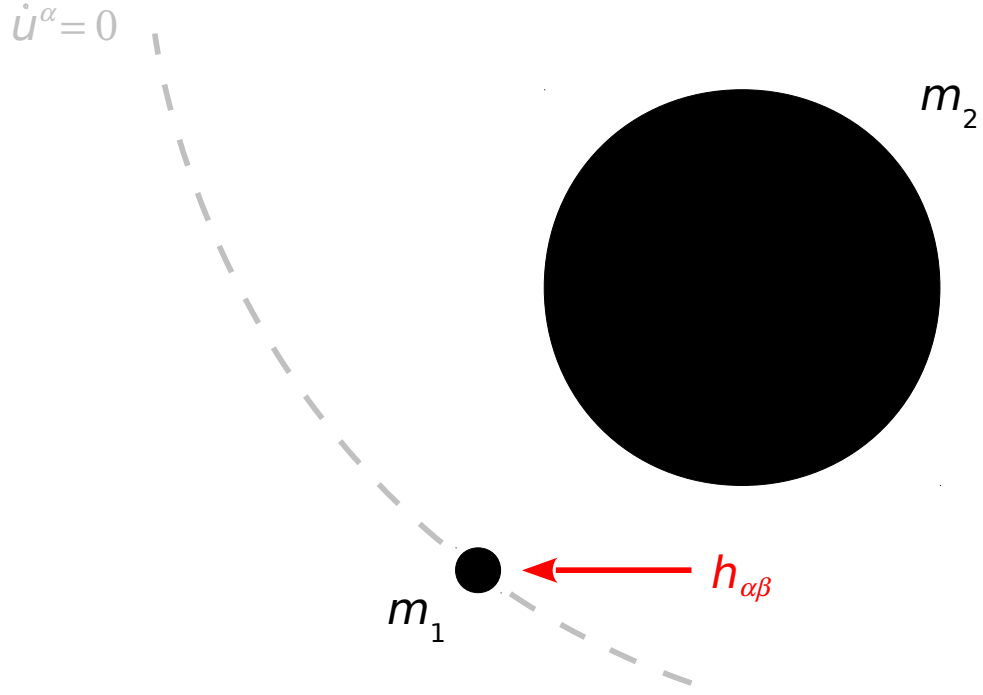

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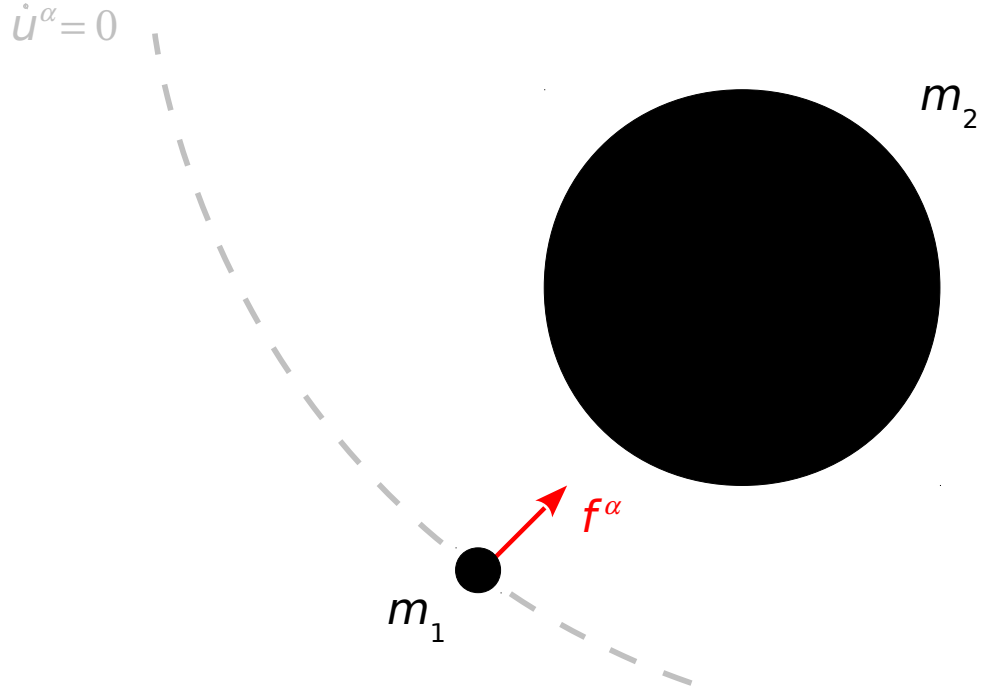

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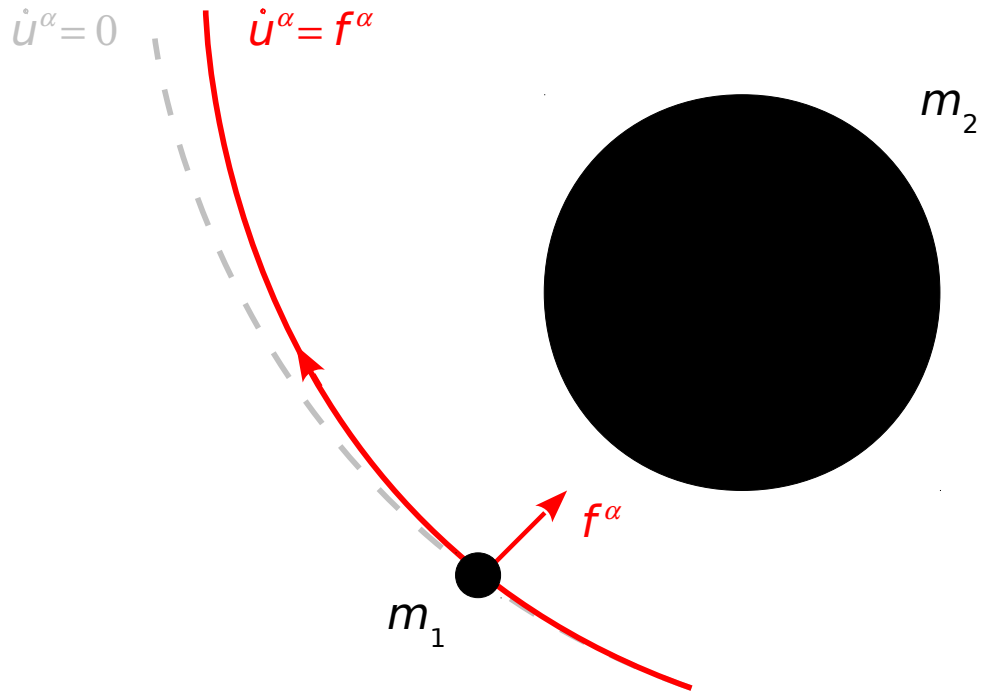
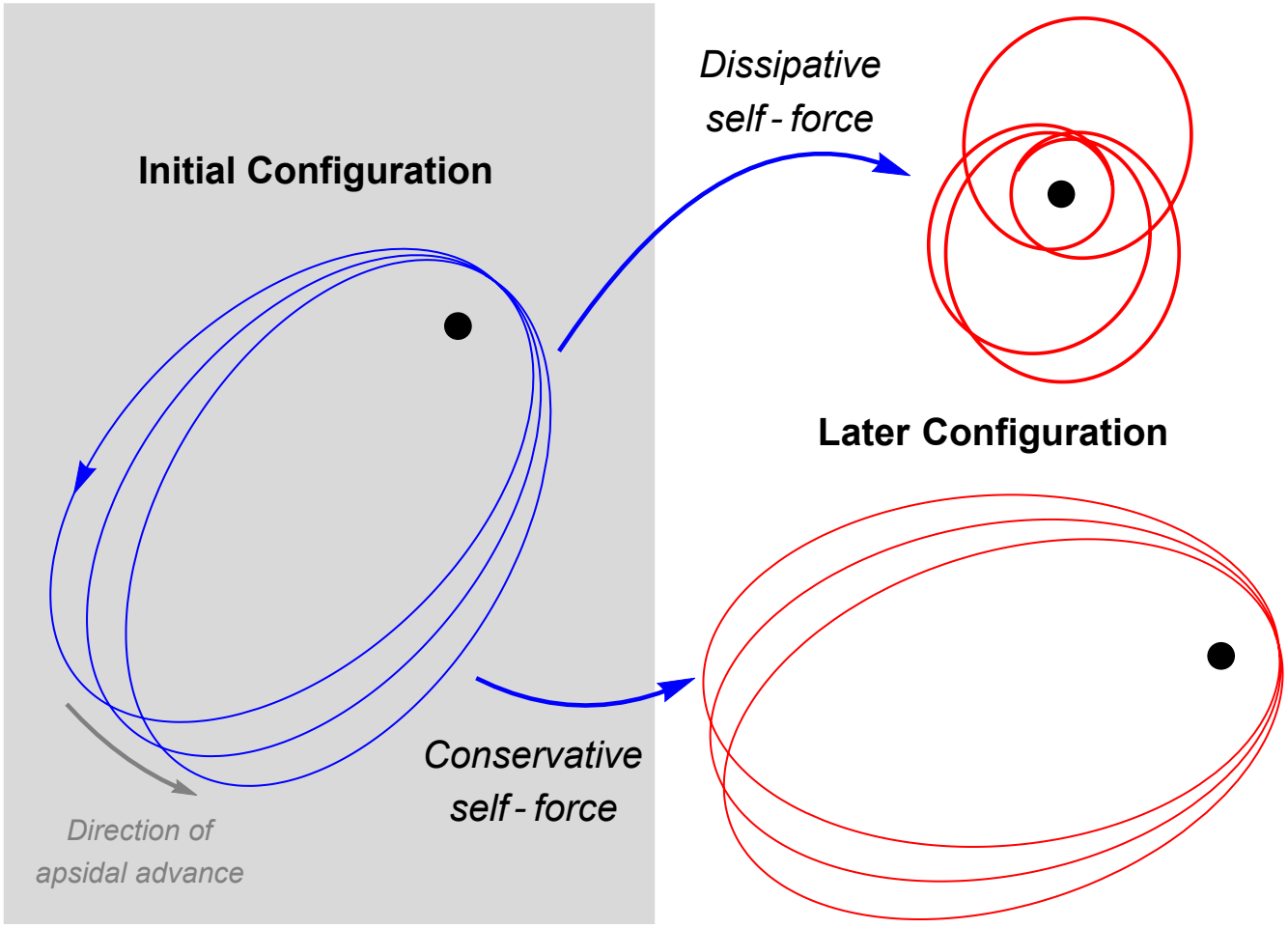
Gravitational self-force

$$\dot{u}^\alpha \equiv u^\beta \mathring{\nabla}_\beta u^\alpha = f^\alpha[h]$$

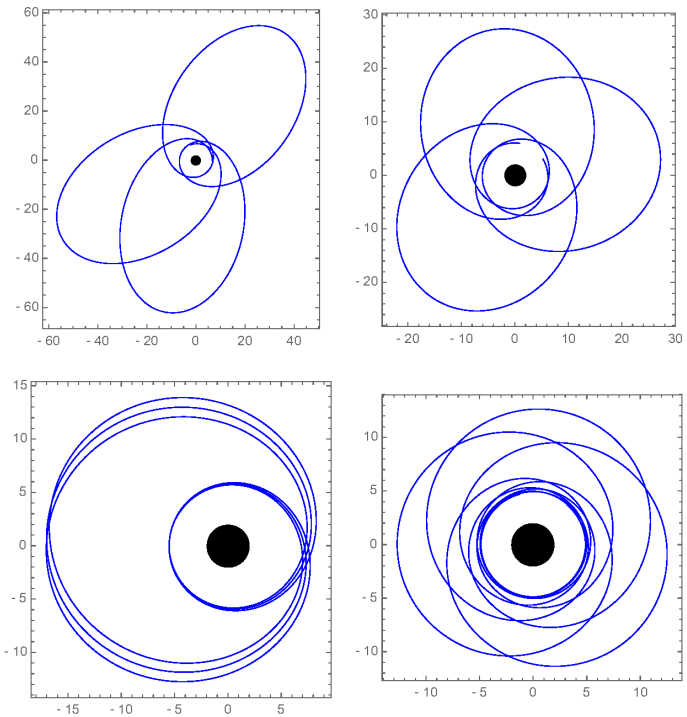


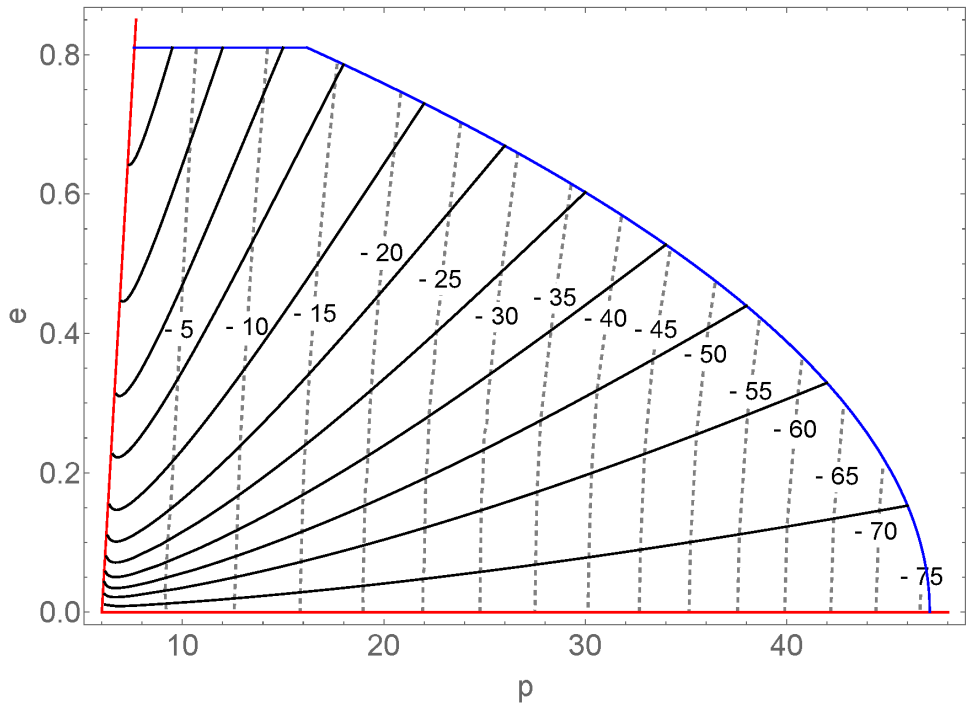
Conservative and dissipative effects



(Credit: Osburn *et al.* 2016)

Inspiral into a non-spinning black hole





[PRD **93** (2016) 064024]

State of the art

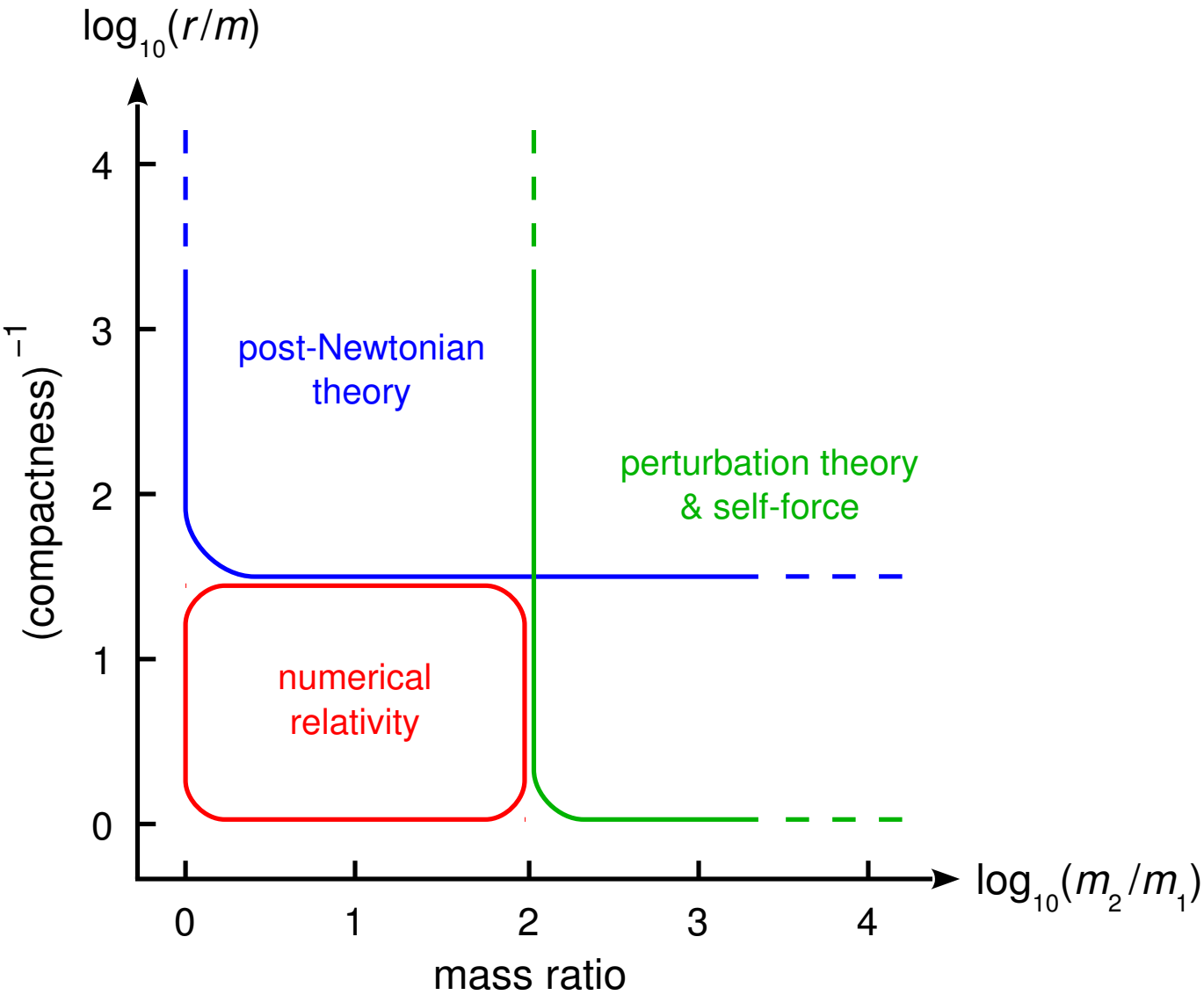
		Adiabatic	1st order	2nd order
Schw.	circular	✔	✔	ongoing
	generic	✔	✔	
	circular	✔	✔	
Kerr	equatorial	✔	✔	
	generic	✔	ongoing	**goal**

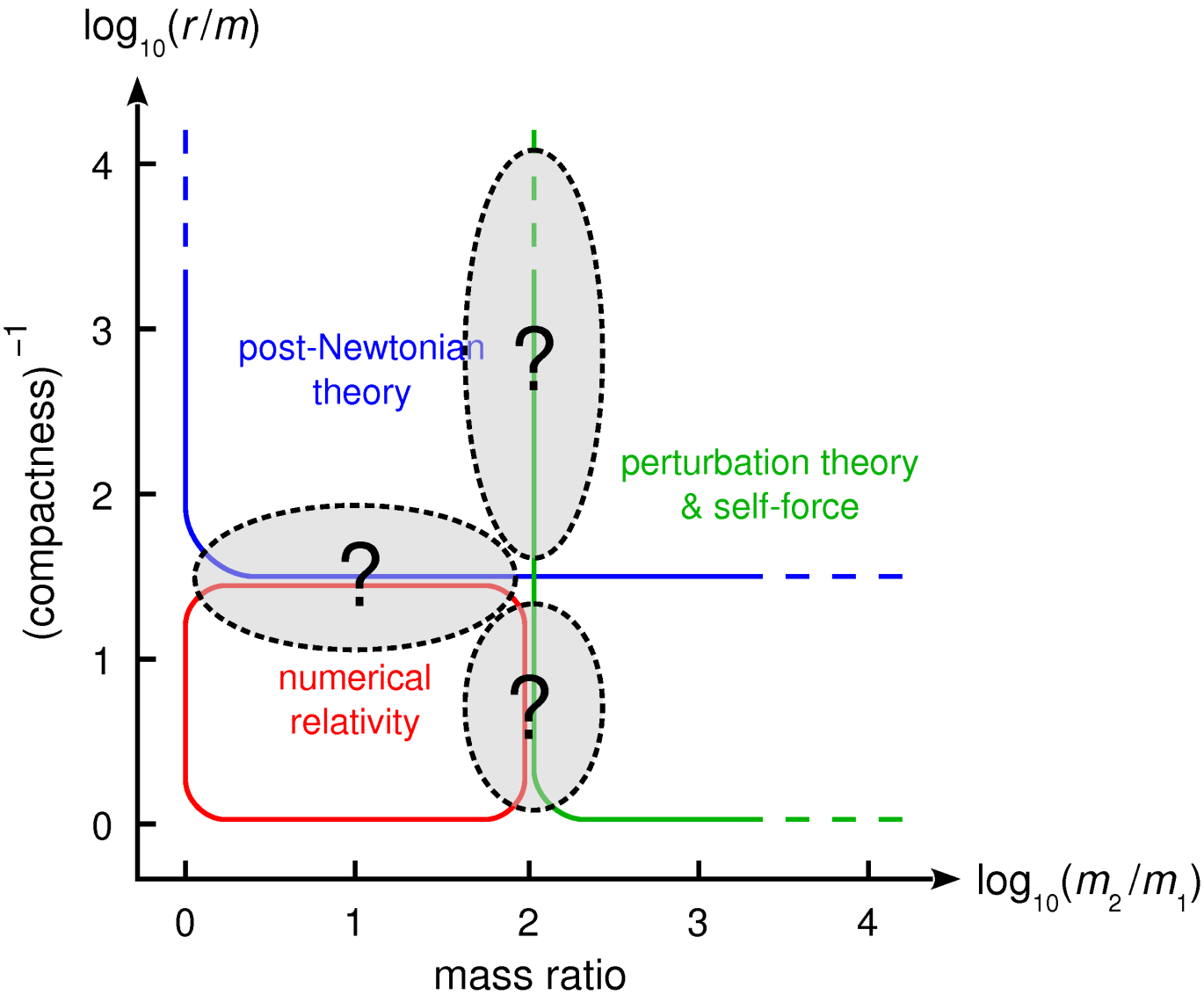
Capra Meetings

19th Capra Meeting on Radiation Reaction (July 2016, Meudon, France)

Outline

1. Context
2. Source modelling
3. EMRIs
4. Comparisons


log10(r/m)
(compactness)−1
4
3
2
1
0
post-Newtonian
theory
perturbation theory
& self-force
numerical
relativity
0
1
2
3
4
log10(m2/m1)
mass ratio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Why?

- Independent checks of long and complicated calculations
- Identify domains of validity of approximation schemes
- Extract information inaccessible to other methods
- Develop a universal model for compact binaries

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✔ Using coordinate-invariant relationships

What?

- Gravitational waveforms at future null infinity
- Conservative effects on the orbital dynamics

Paper	**Year**	**Methods**	**Observable**	**Orbit**	**Spin**
Baker *et al.*	2007	NR/PN	waveform		
Boyle *et al.*	2007	NR/PN	waveform		
Hannam *et al.*	2007	NR/PN	waveform		
Boyle *et al.*	2008	NR/PN/EOB	energy flux		
Damour & Nagar	2008	NR/EOB	waveform		
Hannam *et al.*	2008	NR/PN	waveform		✓
Pan *et al.*	2008	NR/PN/EOB	waveform		
Campanelli *et al.*	2009	NR/PN	waveform		✓
Hannam *et al.*	2010	NR/PN	waveform		✓
Hinder *et al.*	2010	NR/PN	waveform	eccentric	
Lousto *et al.*	2010	NR/BHP	waveform		
Sperhake *et al.*	2011	NR/PN	waveform		
Sperhake *et al.*	2011	NR/BHP	waveform	head-on	
Lousto & Zlochower	2011	NR/BHP	waveform		
Nakano *et al.*	2011	NR/BHP	waveform		
Lousto & Zlochower	2013	NR/PN	waveform		
Nagar	2013	NR/BHP	recoil velocity		
Hinder *et al.*	2014	NR/PN/EOB	waveform		✓
Szilagyi *et al.*	2015	NR/PN/EOB	waveform		
Ossokine *et al.*	2015	NR/PN	waveform		✓

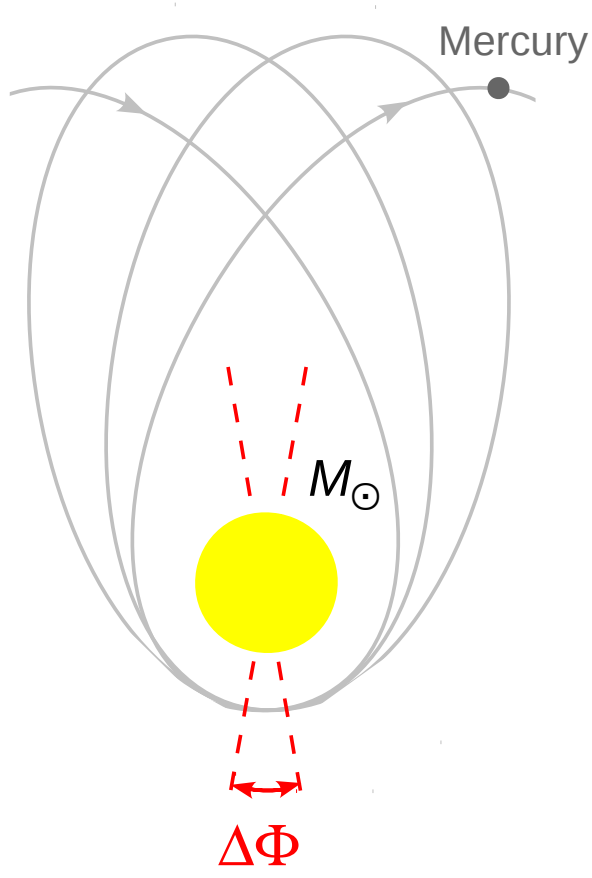
Paper	Year	Methods	Observable	Orbit	Spin
Detweiler	2008	BHP/PN	redshift observable		
Blanchet *et al.*	2010	BHP/PN	redshift observable		
Damour	2010	BHP/EOB	ISCO frequency		
Mroué *et al.*	2010	NR/PN	periastron advance		
Barack *et al.*	2010	BHP/EOB	periastron advance		
Favata	2011	BHP/PN/EOB	ISCO frequency		
Le Tiec *et al.*	2011	NR/BHP/PN/EOB	periastron advance		
Damour *et al.*	2012	NR/EOB	binding energy		
Le Tiec *et al.*	2012	NR/BHP/PN/EOB	binding energy		
Akcay *et al.*	2012	BHP/EOB	redshift observable		
Hinderer *et al.*	2013	NR/EOB	periastron advance		✓
Le Tiec *et al.*	2013	NR/BHP/PN	periastron advance		✓
Bini & Damour Shah *et al.* Blanchet *et al.*	2014	BHP/PN	redshift observable		
Dolan *et al.* Bini & Damour	2014	BHP/PN	precession angle		✓
Isoyama *et al.*	2014	BHP/PN/EOB	ISCO frequency		✓
Akcay *et al.*	2015	BHP/PN	averaged redshift	eccentric	
Shah & Pound	2015	BHP/PN	precession angle		✓
Zimmerman *et al.*	2016	NR/PN	surface gravity		
Akcay *et al.*	2016	BHP/PN	precession angle	eccentric	

Relativistic perihelion advance of Mercury

- Observed anomalous advance of Mercury's perihelion of $\sim 43''$/cent.
- Accounted for by the leading-order relativistic angular advance per orbit

$$\Delta\Phi = \frac{6\pi G M_\odot}{c^2 a\,(1-e^2)}$$

- Periastron advance of $\sim 4^\circ$/yr now measured in **binary pulsars**



Periastron advance in black hole binaries

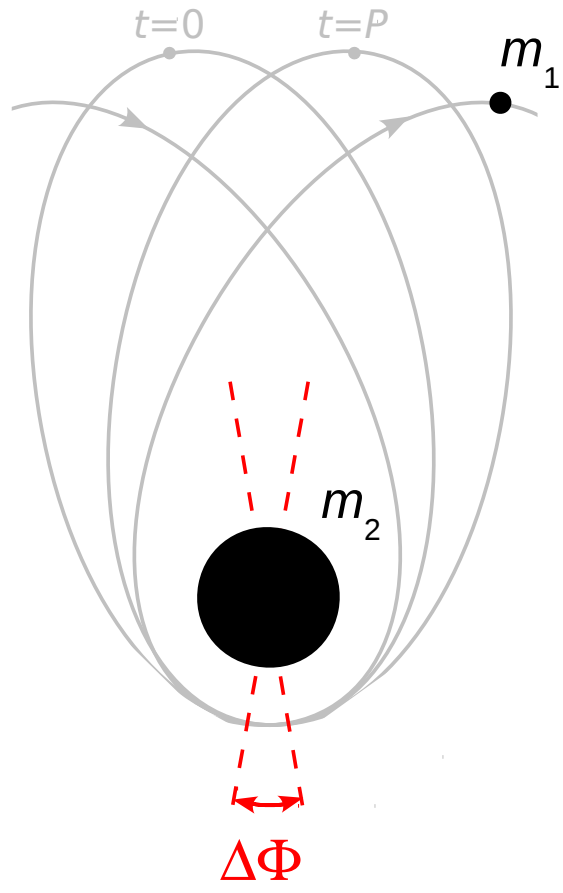
- Generic eccentric orbit parametrized by the two invariant frequencies

$$\Omega_r = \frac{2\pi}{P}\,, \quad \Omega_\varphi = \frac{1}{P}\int_0^P \dot{\varphi}(t)\,\mathrm{d}t$$

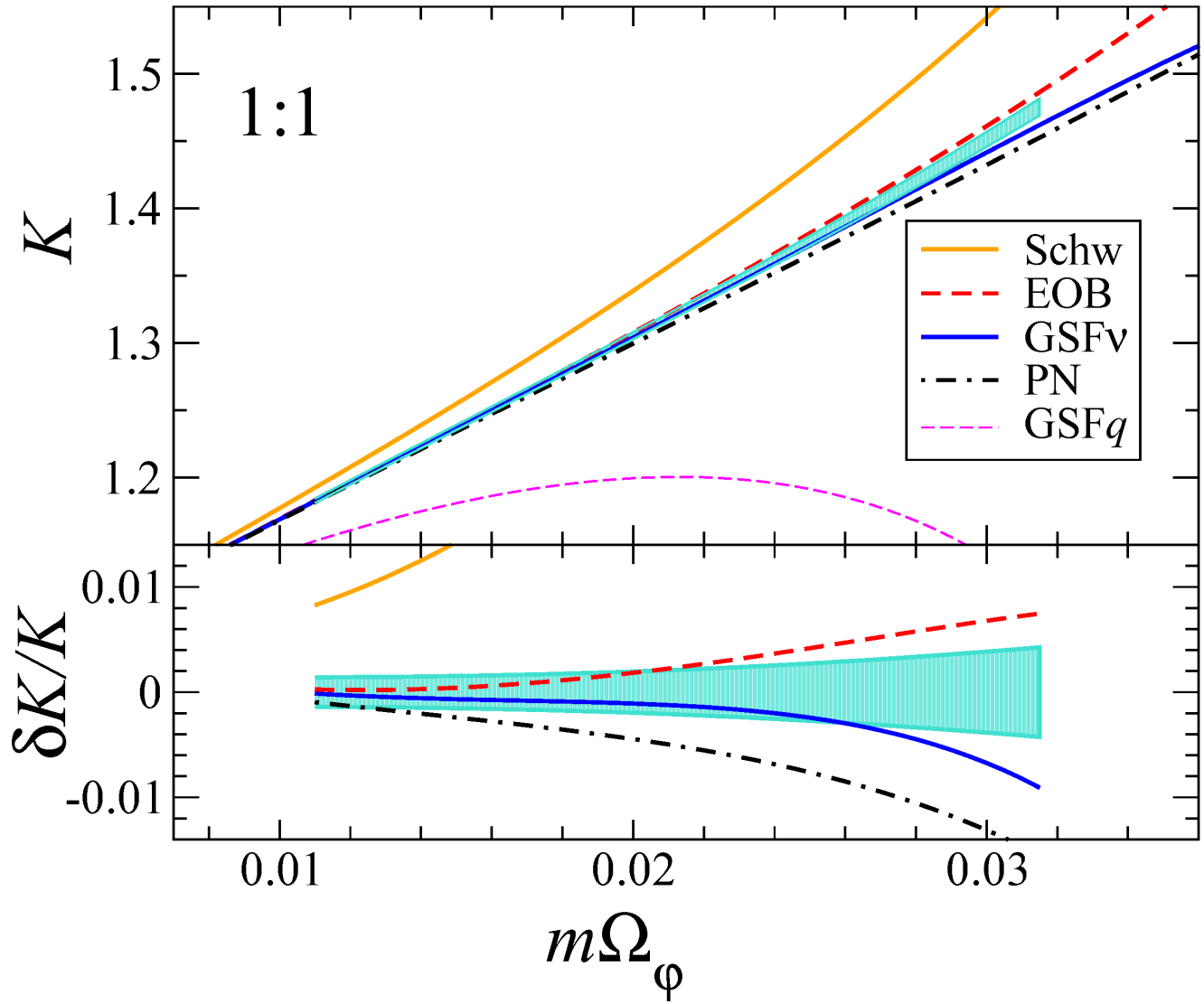
- Periastron advance per radial period

$$K \equiv \frac{\Omega_\varphi}{\Omega_r} = 1 + \frac{\Delta\Phi}{2\pi}$$

- In the circular orbit limit $e \to 0$, the relation $K(\Omega_\varphi)$ is coordinate-invariant

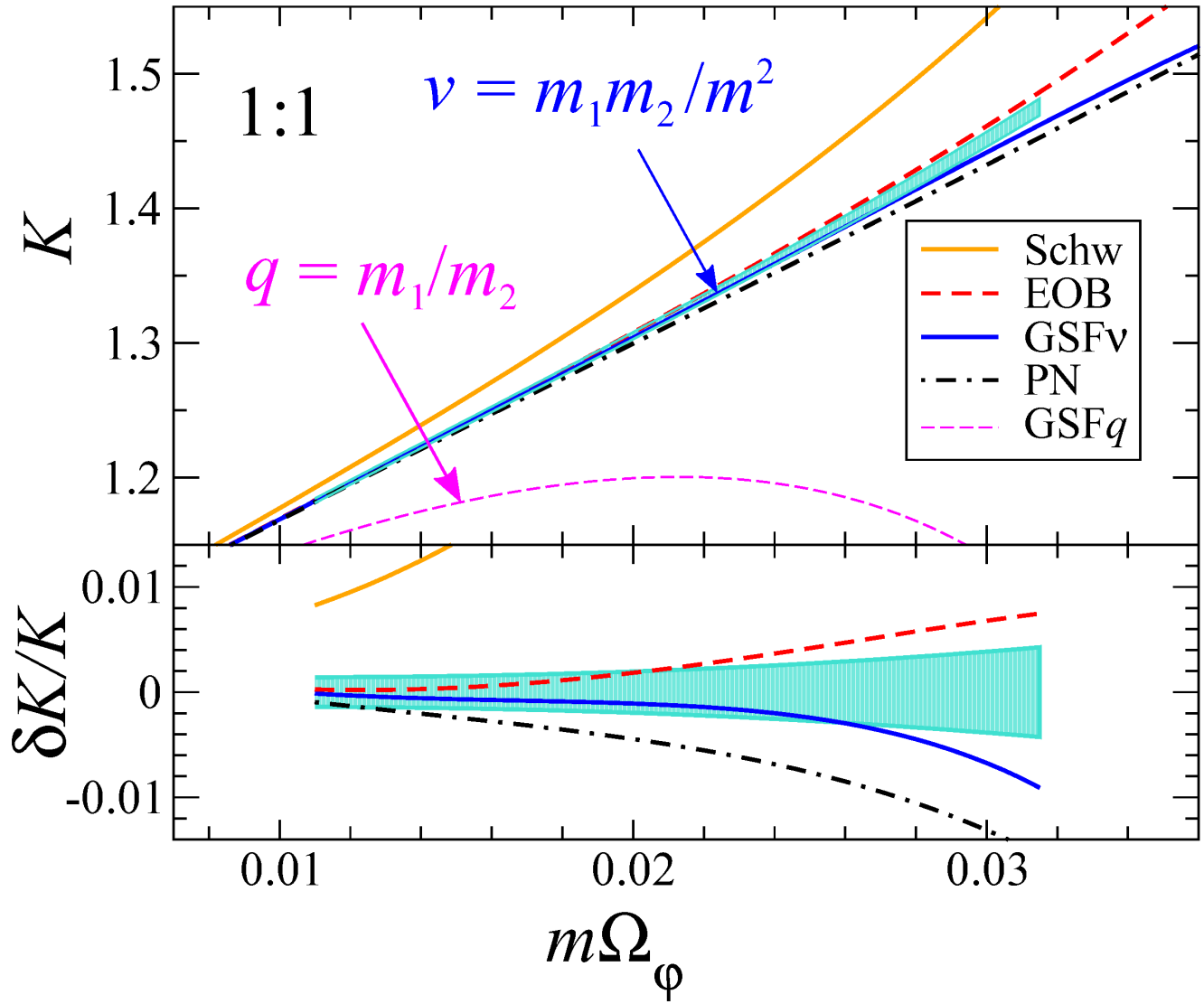


Periastron advance *vs* orbital frequency



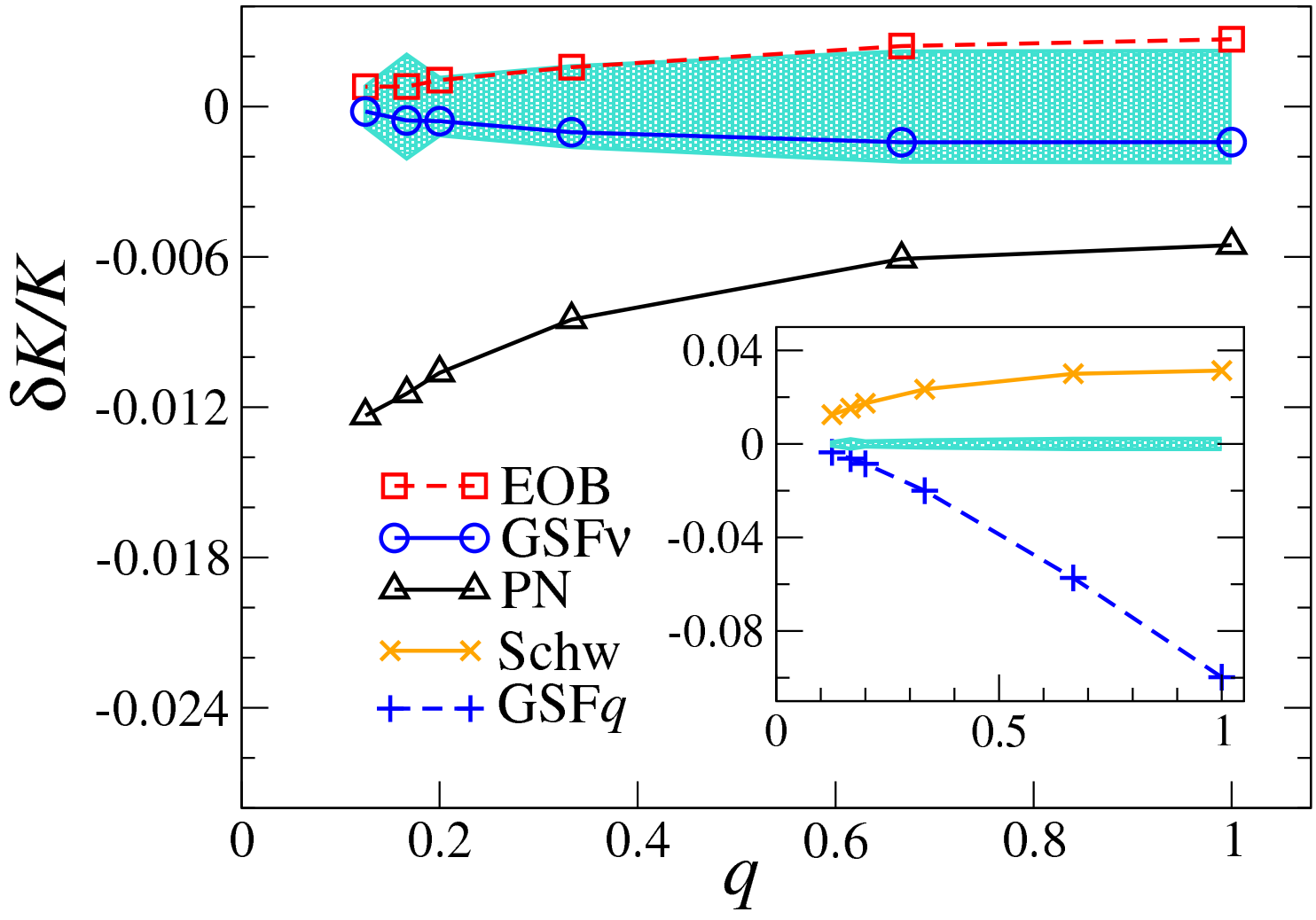
[PRL **107** (2011) 141101]

Periastron advance *vs* orbital frequency



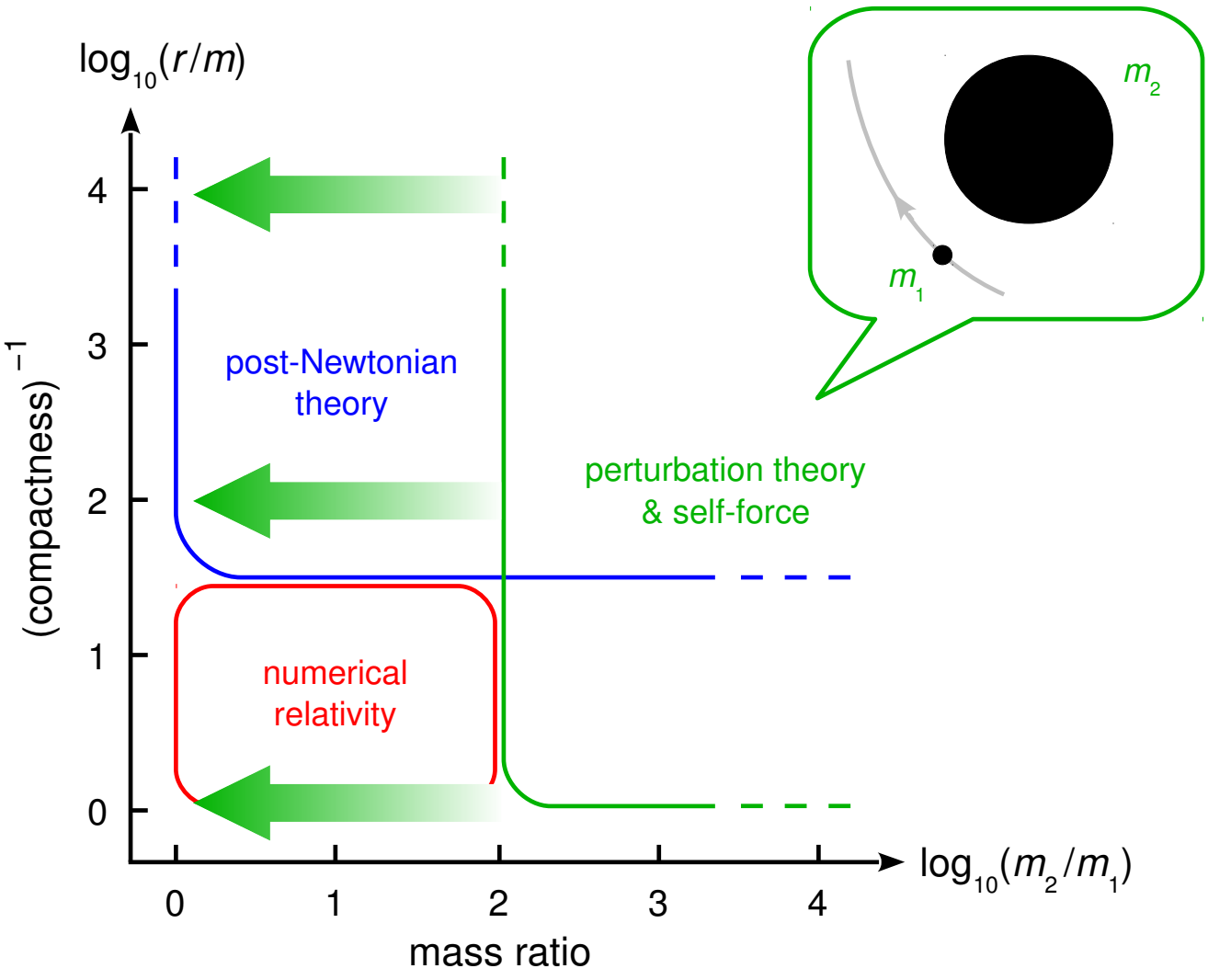
[PRL **107** (2011) 141101]

Periastron advance *vs* mass ratio

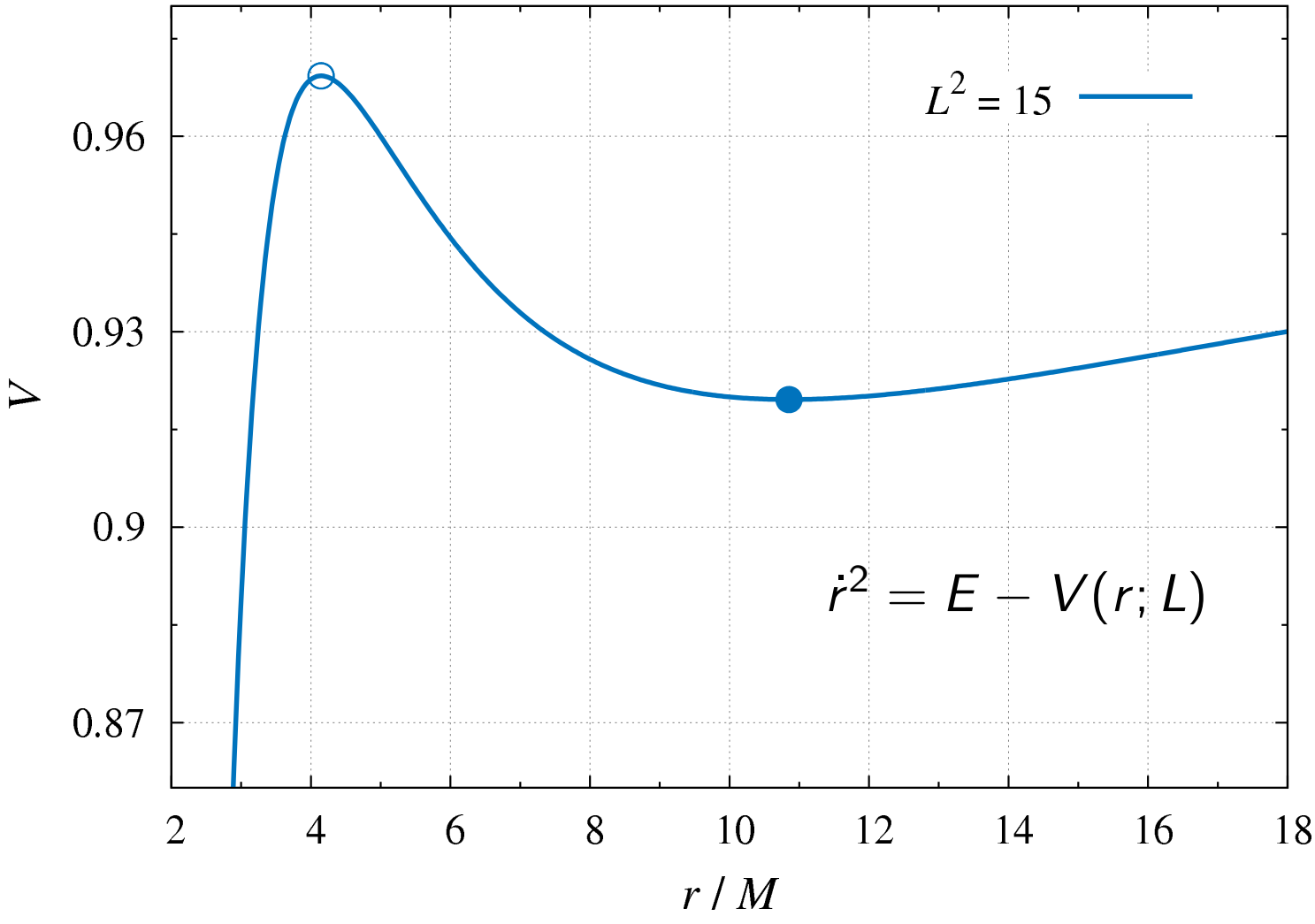


[PRL **107** (2011) 141101]

Perturbation theory for comparable masses



Innermost stable circular orbit (ISCO)



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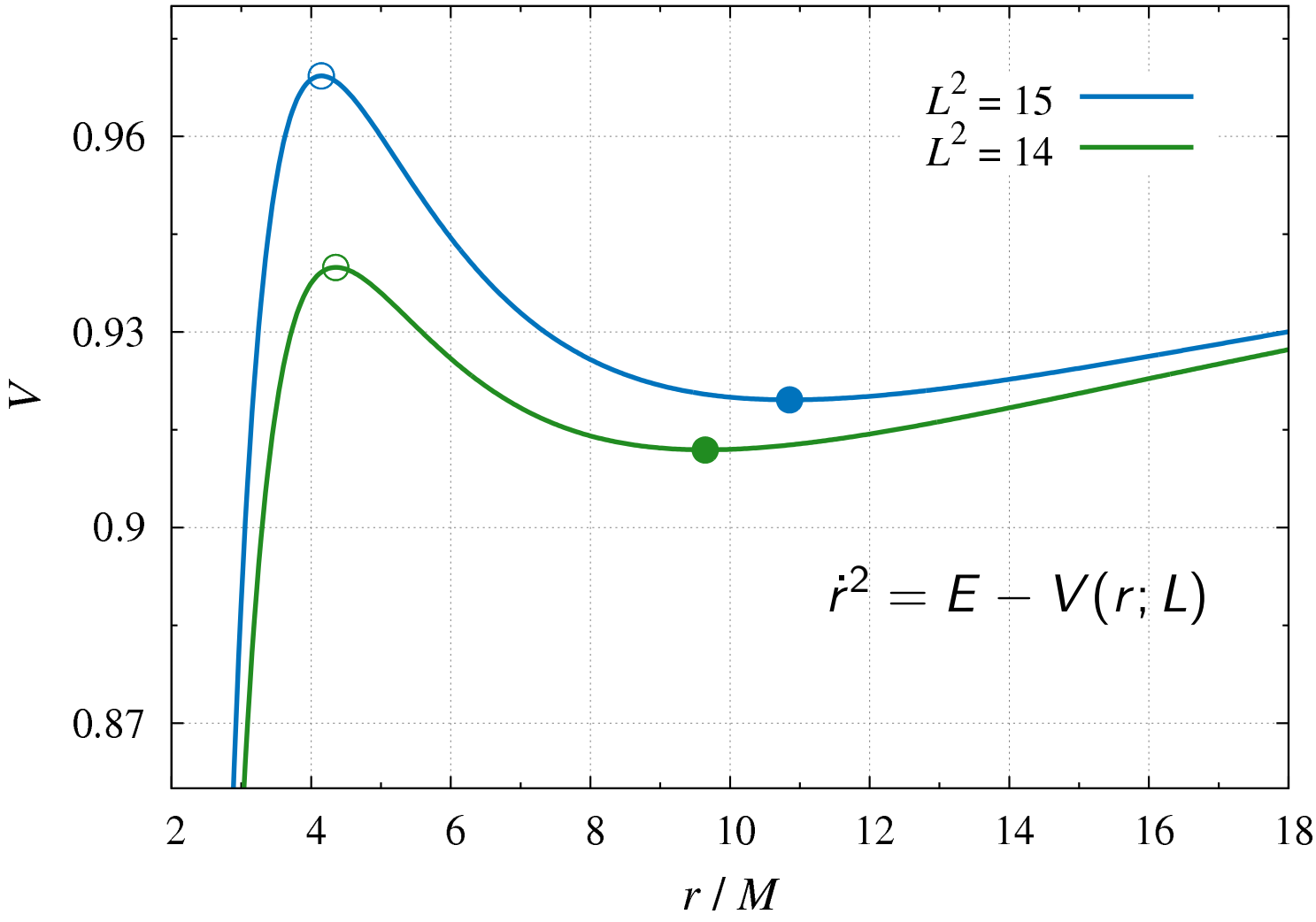

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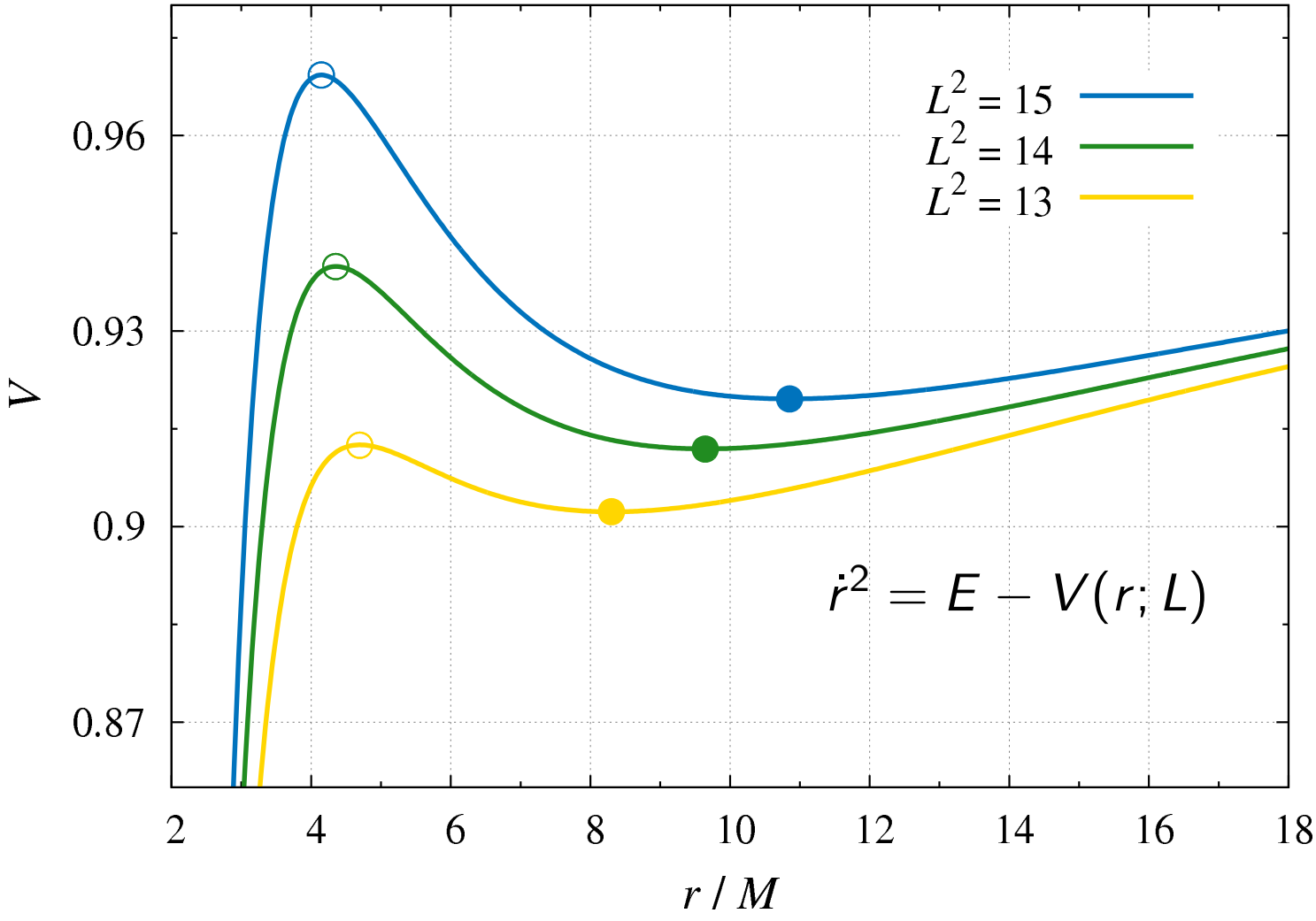

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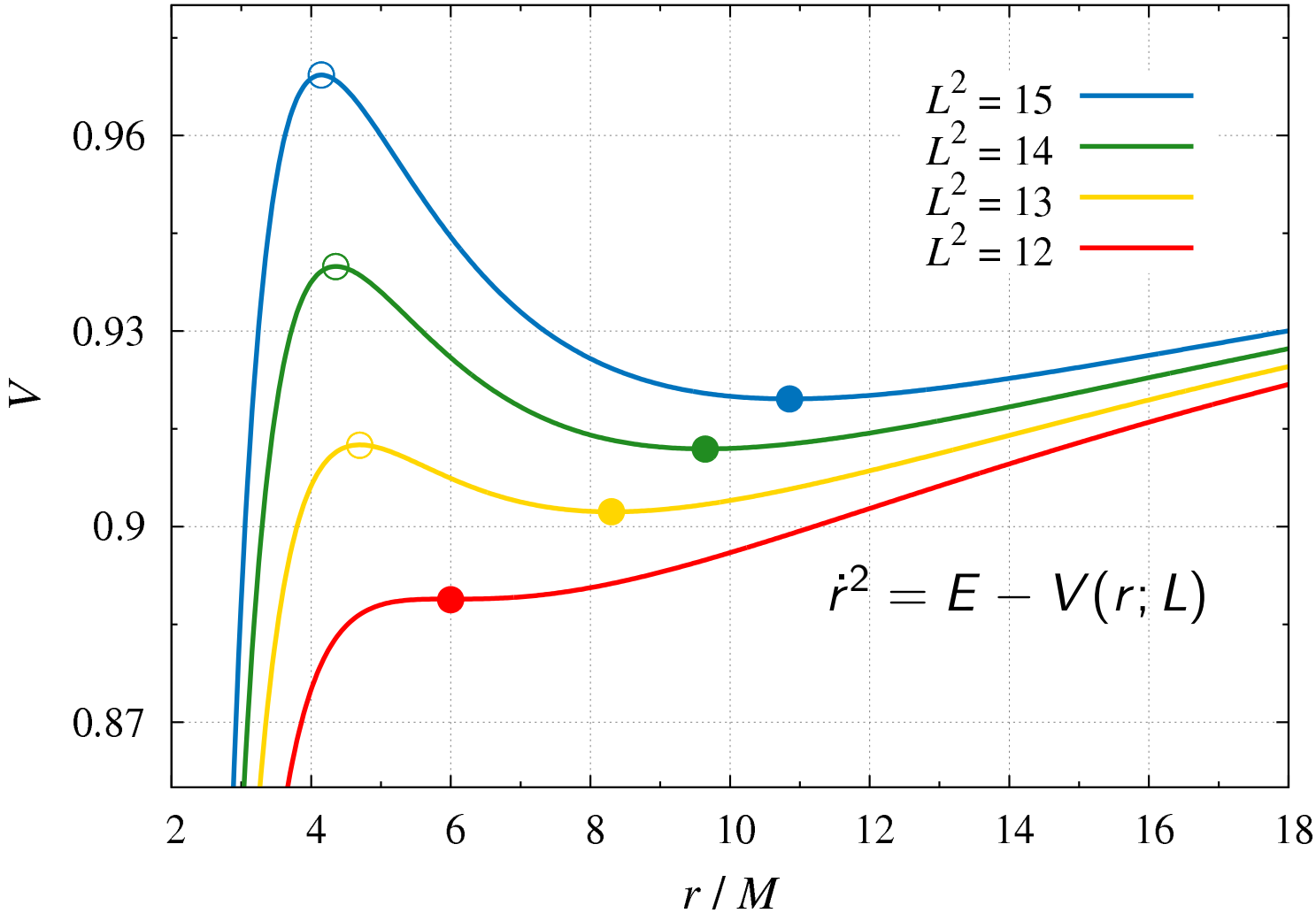


Innermost stable circular orbit (ISCO)

- The innermost stable circular orbit is identified by a vanishing restoring radial force under small-e perturbations:

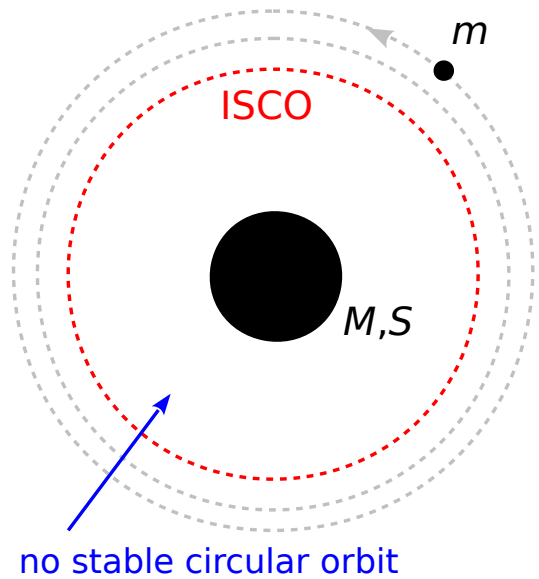
$$\frac{\partial^2 H}{\partial r^2} = 0 \quad \longrightarrow \quad \Omega_{\rm isco}$$

- The minimum energy circular orbit is the most bound orbit along a sequence of circular orbits:

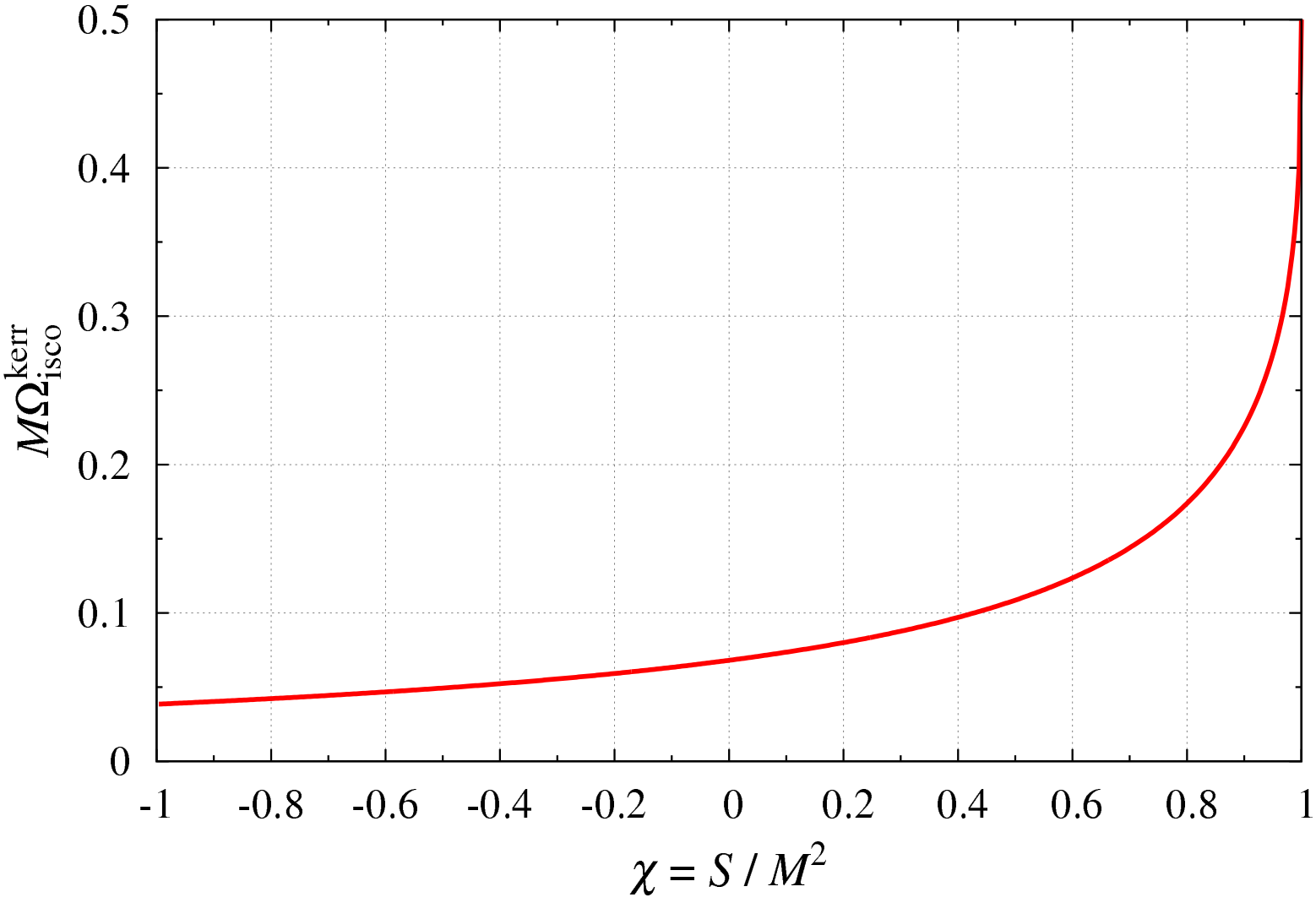
$$\frac{\partial E}{\partial \Omega} = 0 \quad \longrightarrow \quad \Omega_{\rm meco}$$

- For Hamiltonian systems, it can be shown that

$$\Omega_{\rm isco} = \Omega_{\rm meco}$$



Kerr ISCO frequency *vs* black hole spin



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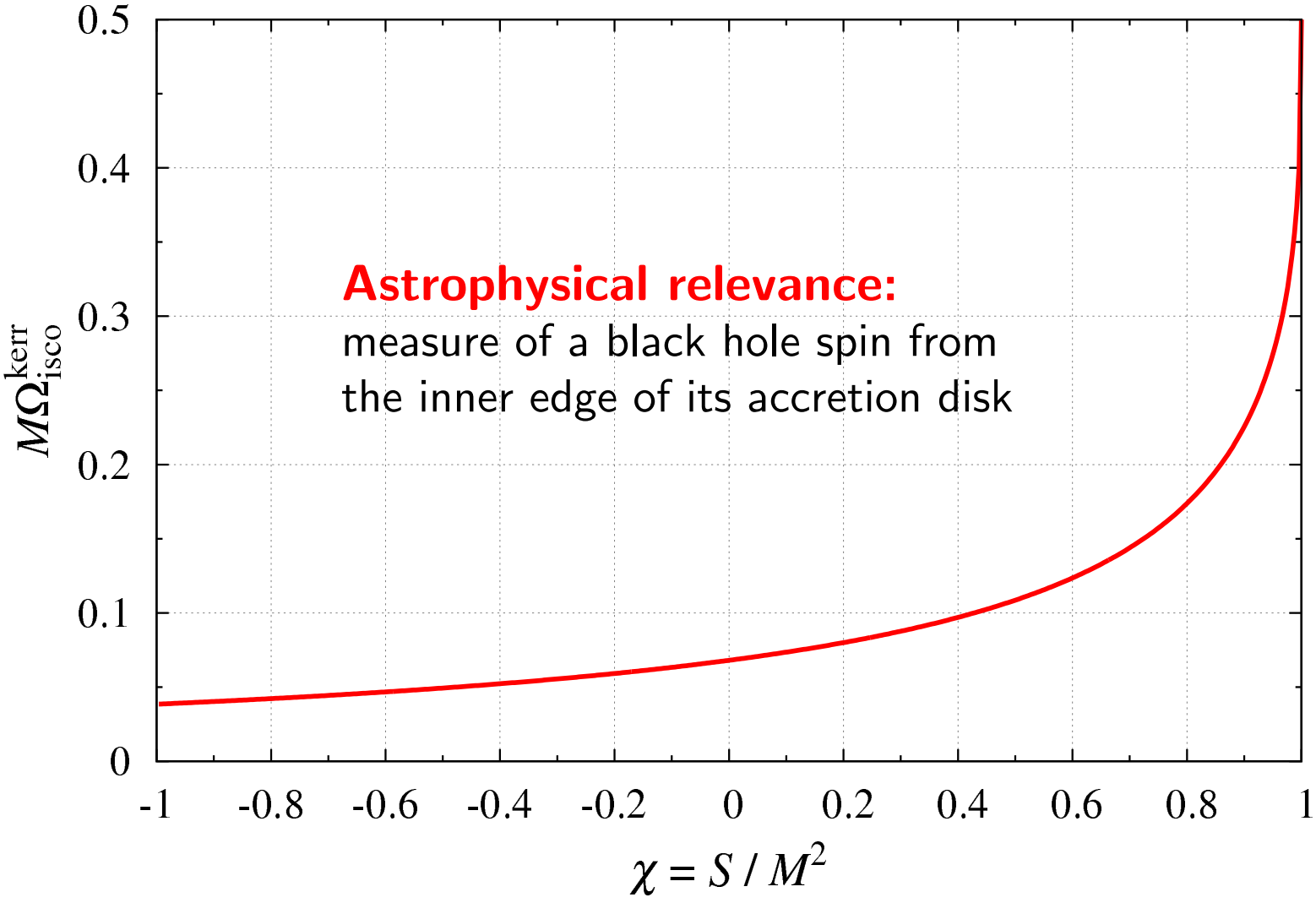
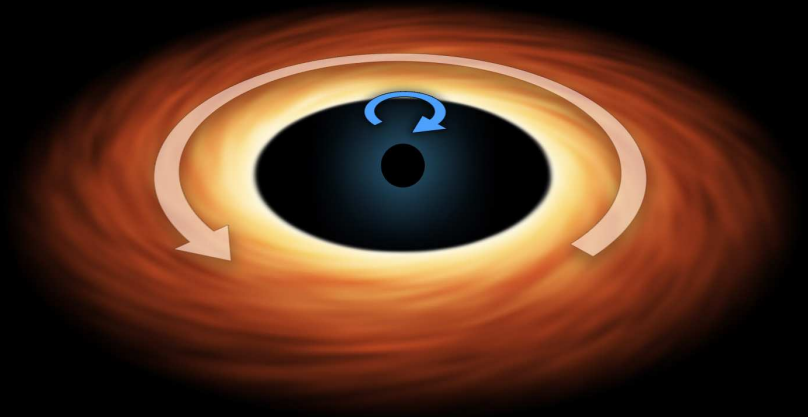
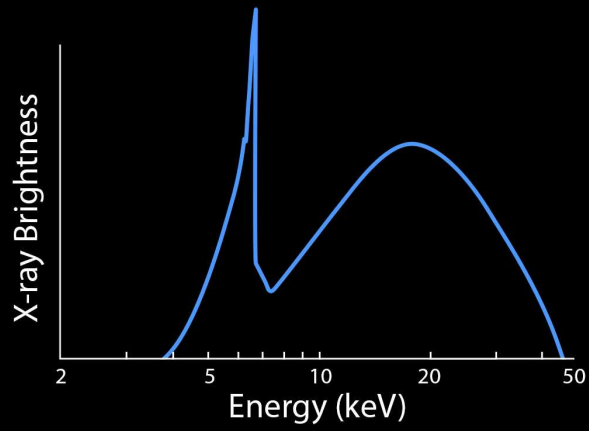
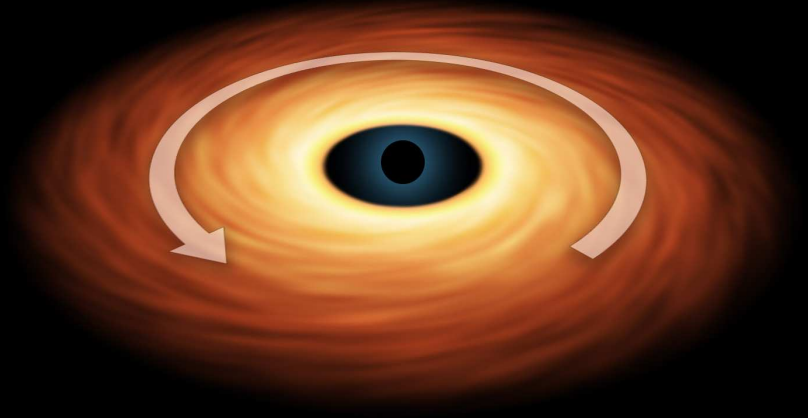


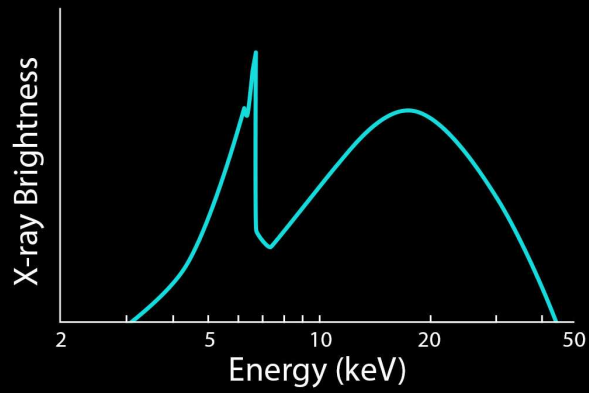


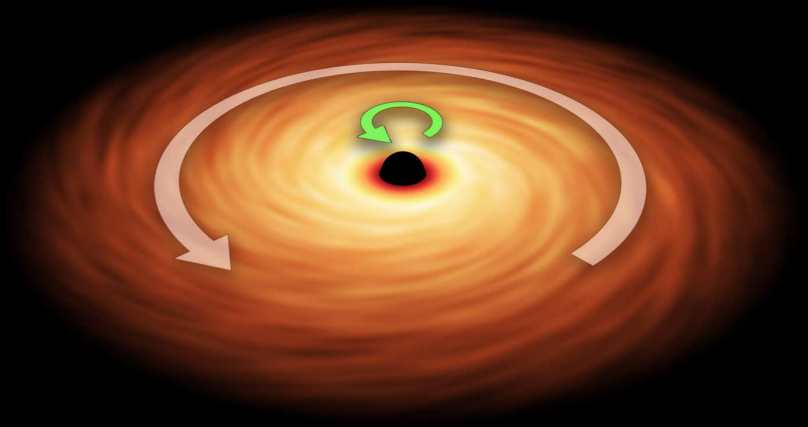
Retrograde Rotation



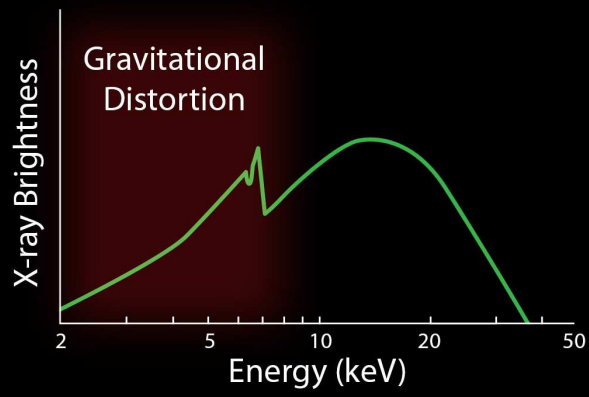


No Black Hole Rotation

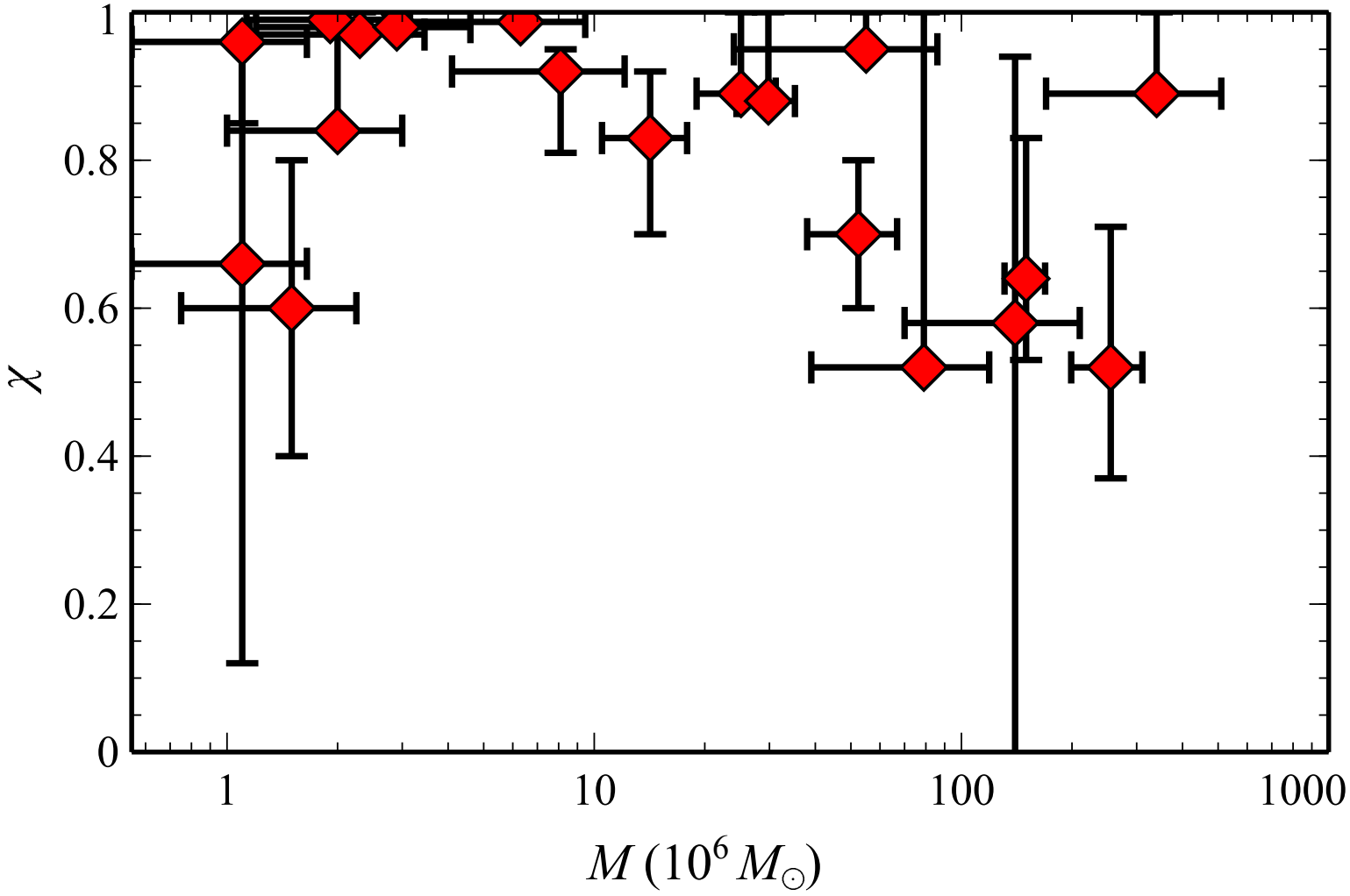




Prograde Rotation



Spins of supermassive black holes



[CQG **30** (2013) 244004]

Frequency shift of the Kerr ISCO

- The orbital frequency of the Kerr ISCO is shifted under the effect of the conservative self-force:

$$(M+m)\Omega_{\rm isco} = \underbrace{M\Omega^{(0)}_{\rm isco}(\chi)}_{\substack{\text{test mass}\\ \text{result}}} \left\{ 1 + \underbrace{q\, C_\Omega(\chi)}_{\substack{\text{self-force}\\ \text{correction}}} + \mathcal{O}(q^2) \right\}$$

[PRL **113** (2014) 161101]

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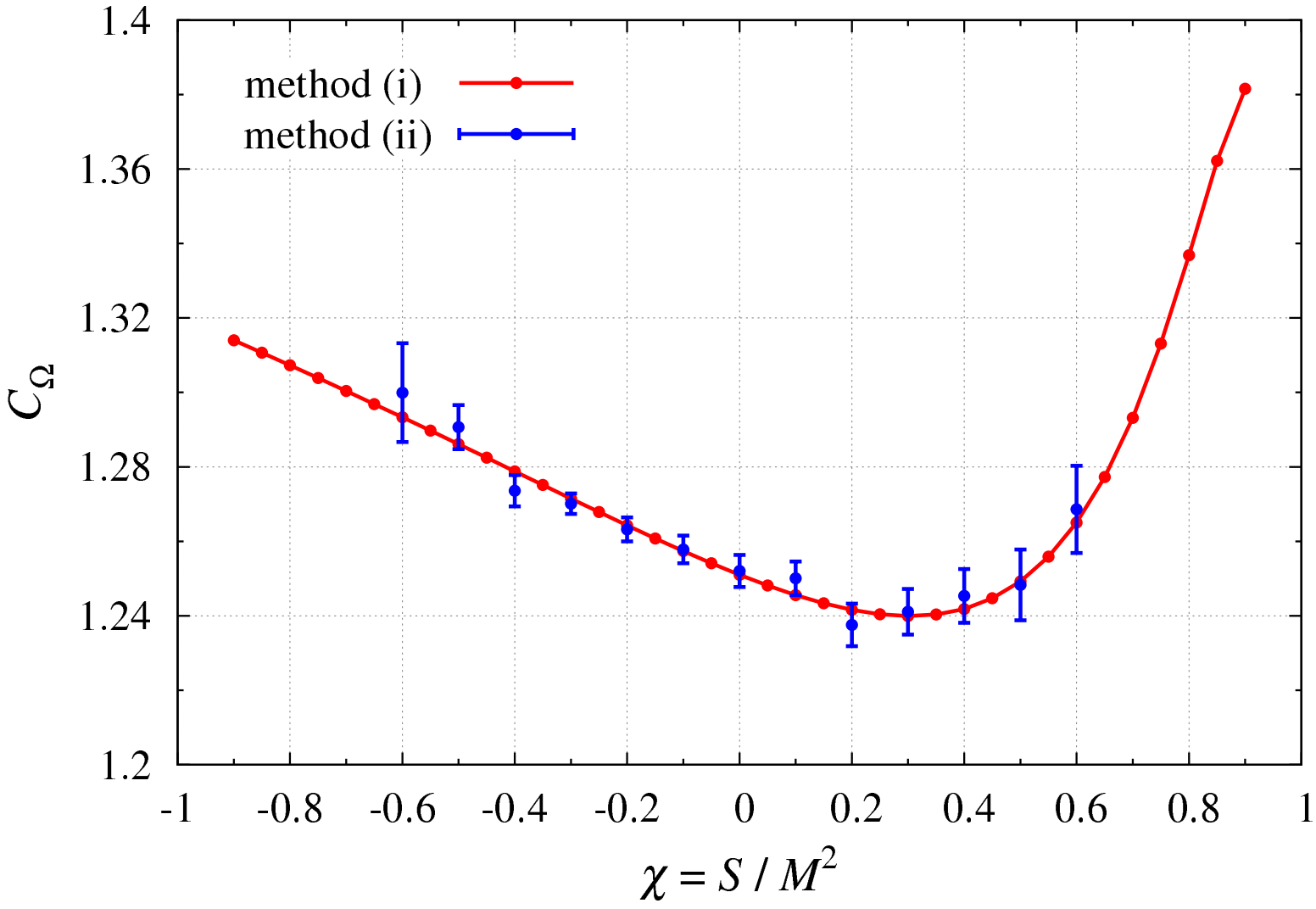
- The frequency shift can be computed from a stability analysis of slightly eccentric orbits near the Kerr ISCO
- Combining the Hamiltonian first law with the MECO conditio $\partial E/\partial\Omega = 0$ yields the same result:

$$C_\Omega = \frac{1}{2}\,\frac{z''_{(1)}(\Omega^{(0)}_{\rm isco})}{E''_{(0)}(\Omega^{(0)}_{\rm isco})}$$

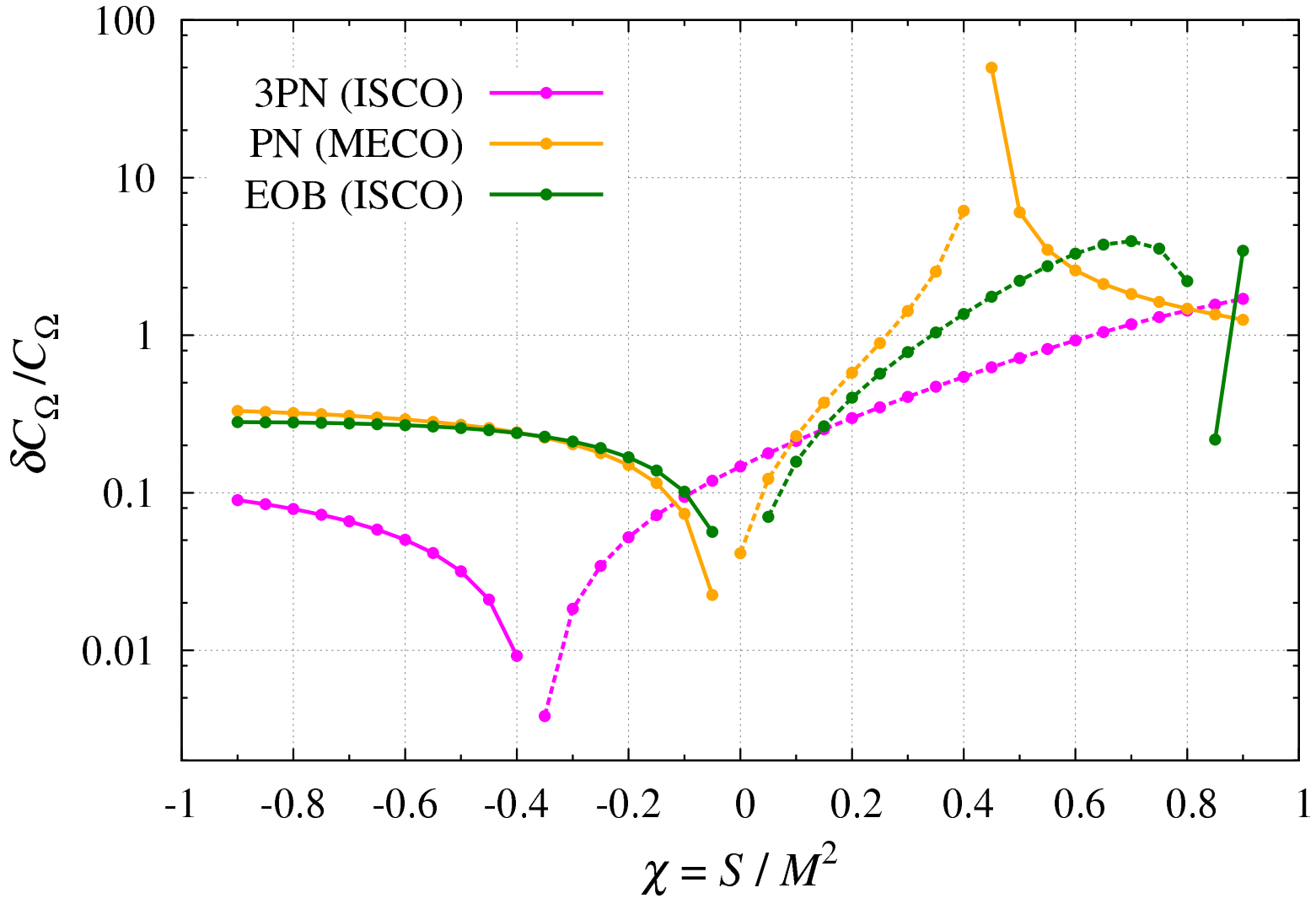
[PRL **113** (2014) 161101]

ISCO frequency shift *vs* black hole spin



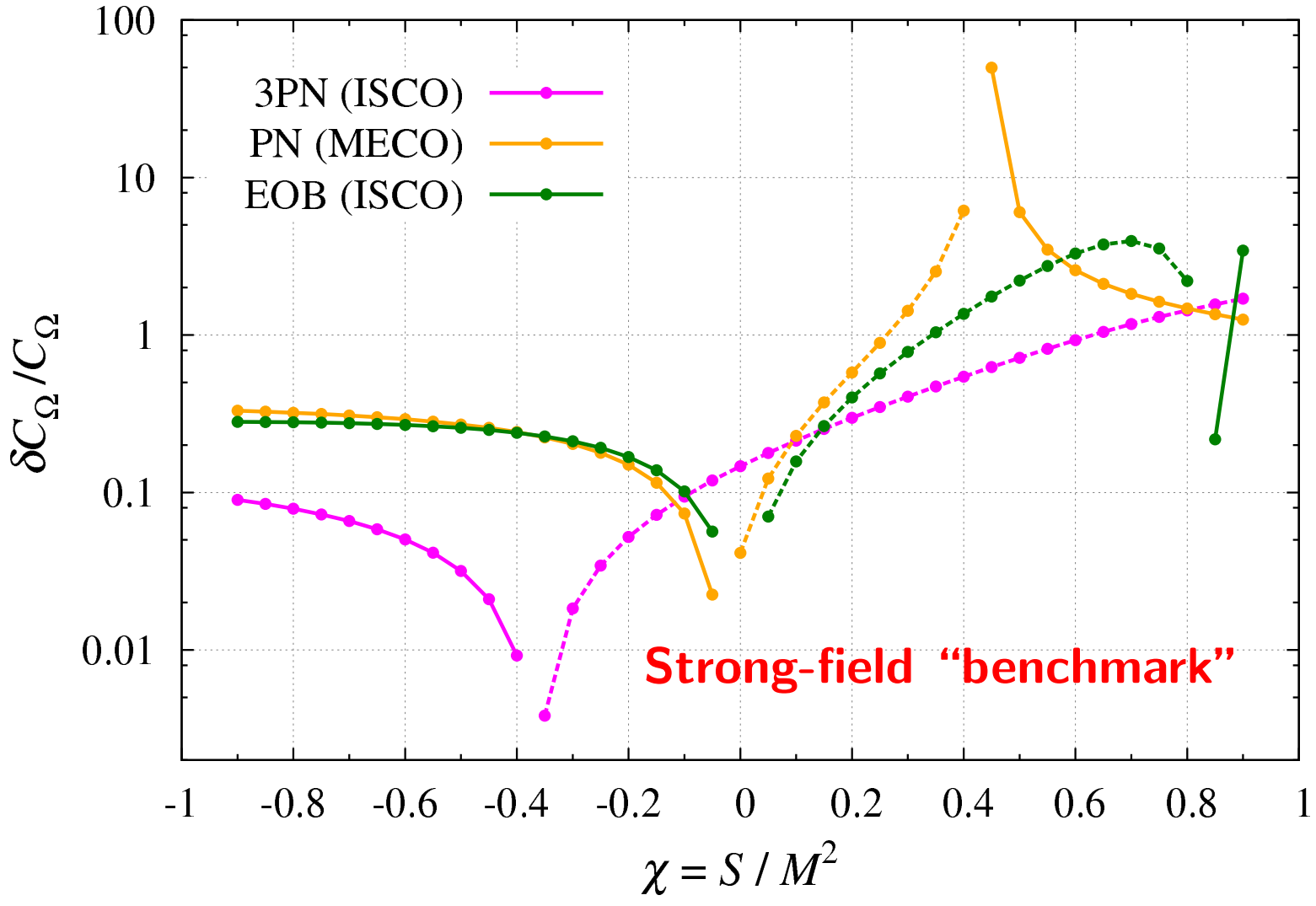
[PRL **113** (2014) 161101]

ISCO frequency shift *vs* black hole spin



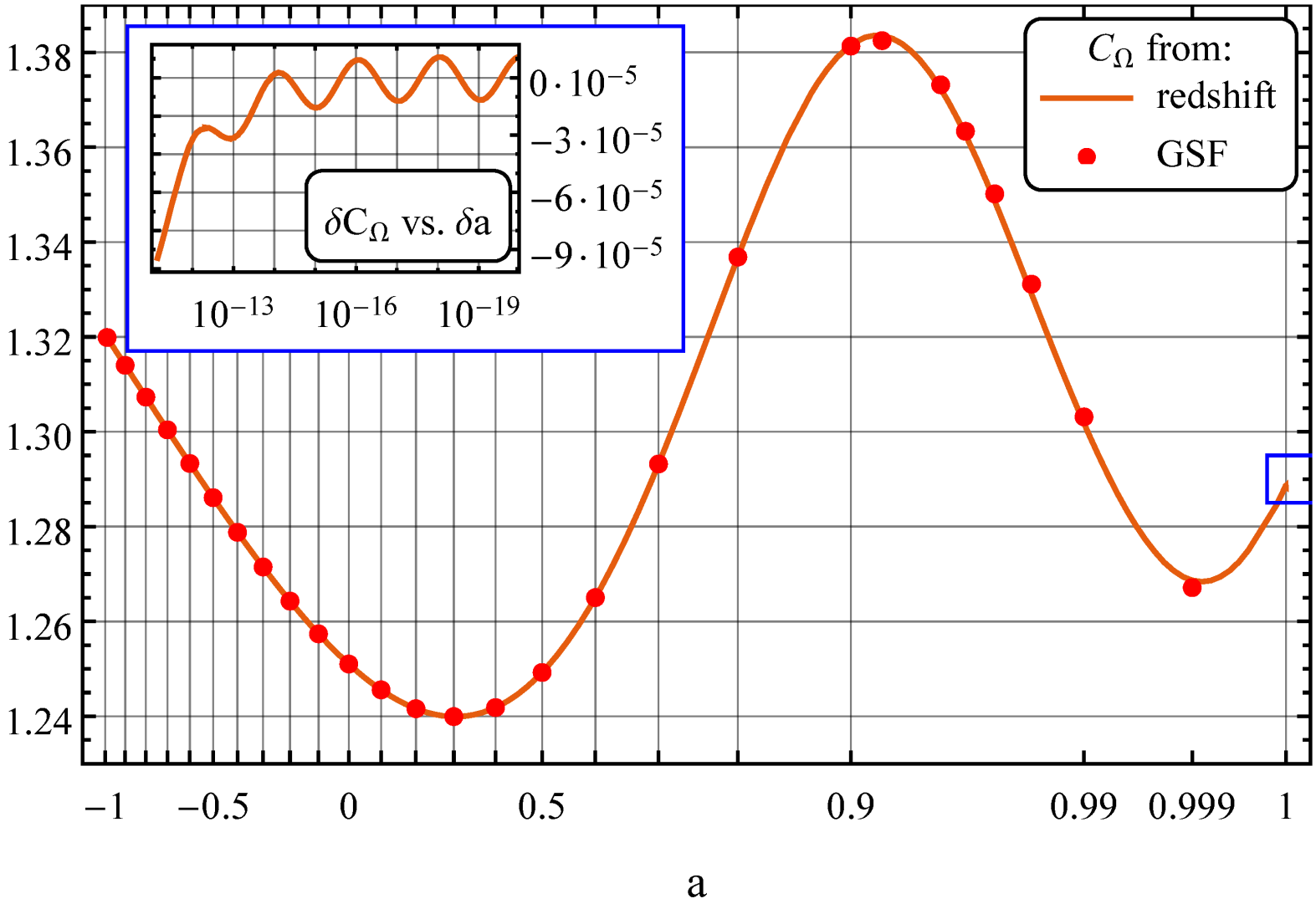
[PRL **113** (2014) 161101]

ISCO frequency shift *vs* black hole spin



[PRL **113** (2014) 161101]

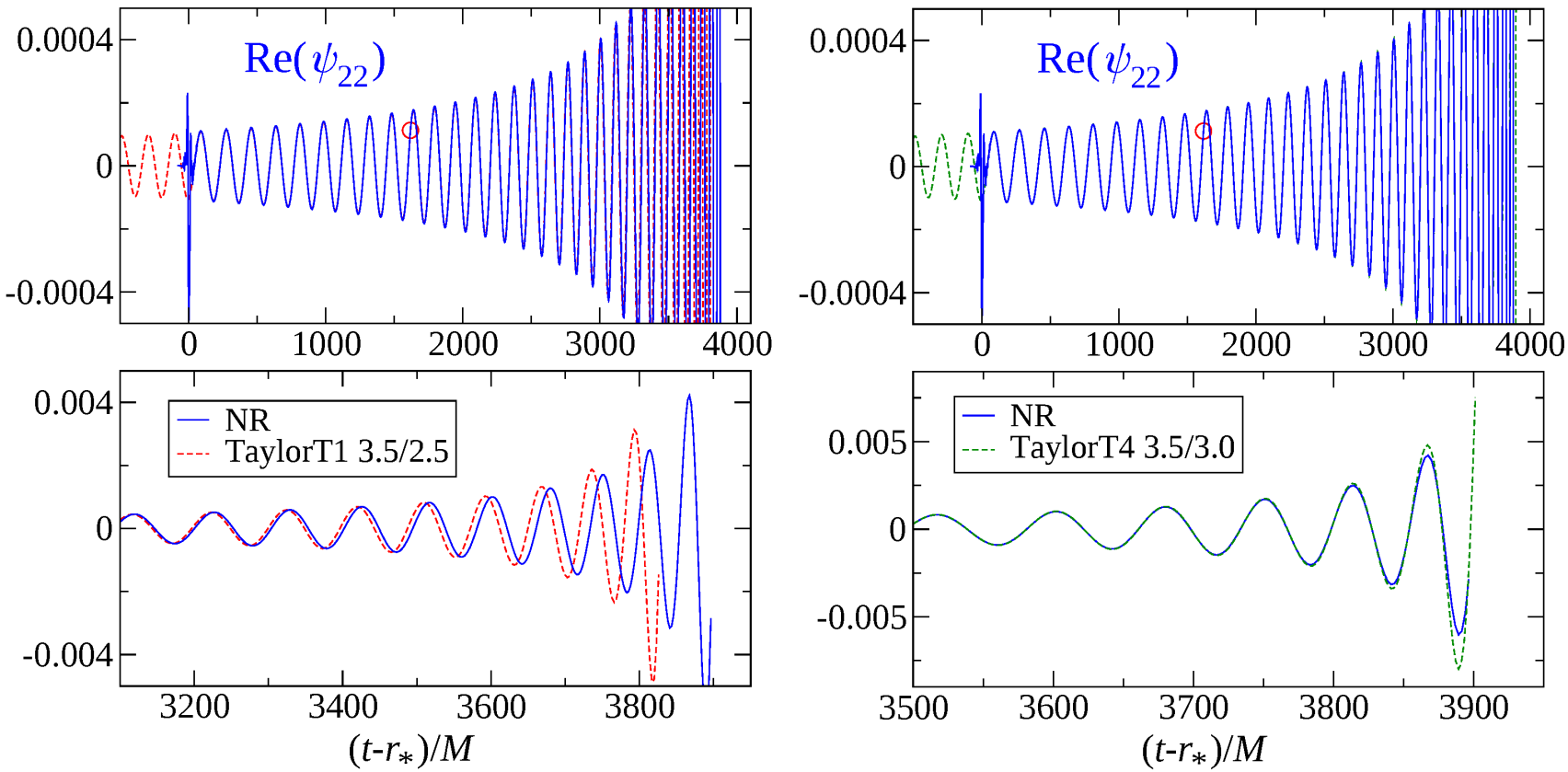
ISCO frequency shift *vs* black hole spin



[PRL **118** (2017) 011101]

PN *vs* NR waveforms

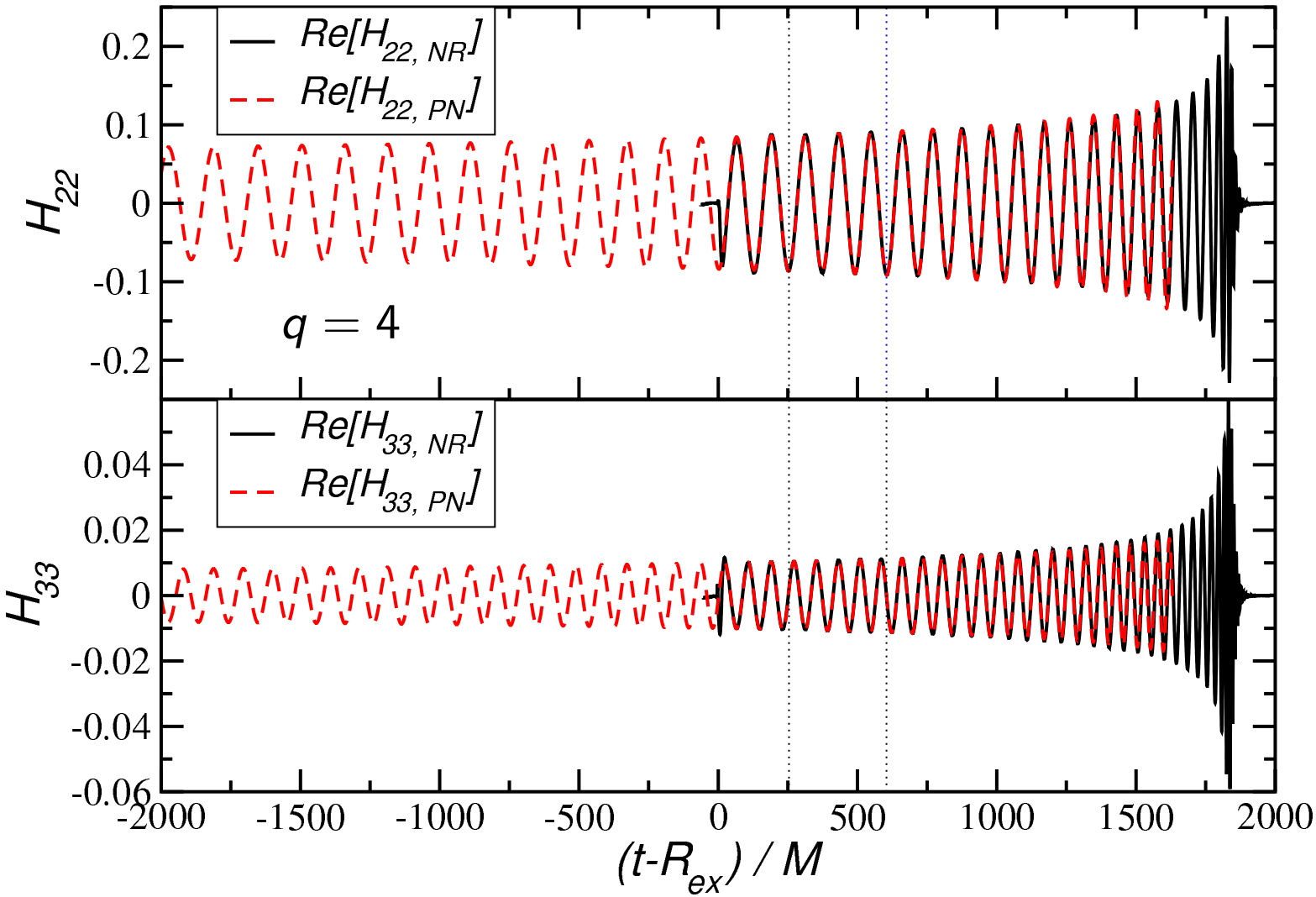
Equal masses and no spins



[PRD **76** (2007) 124038]

PN *vs* NR waveforms

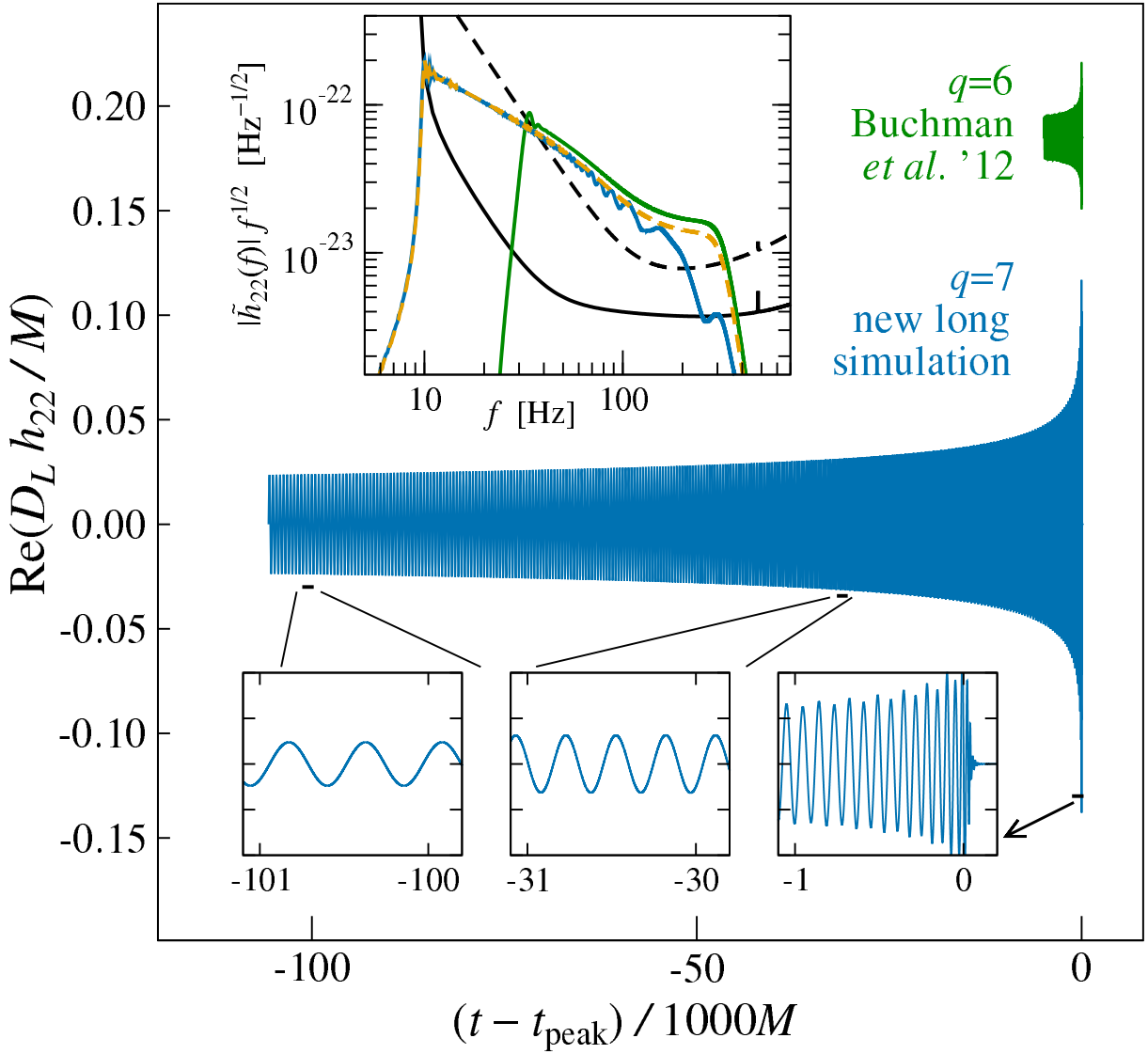
Unequal masses and no spins



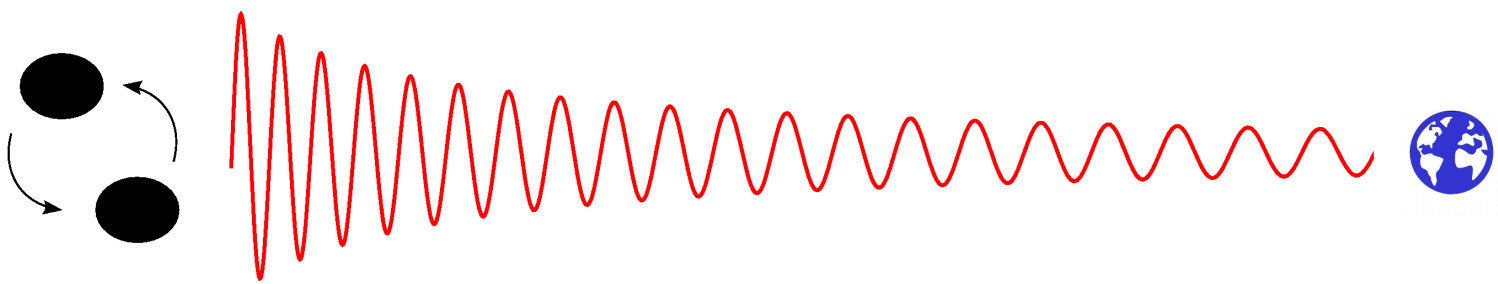
[CQG **28** (2011) 134004]

350 GW cycles!

PN *vs* NR waveforms



[PRL **115** (2015) 031102]

Summary

- The dynamics and wave emission from black hole binaries can be investigated using a variety of methods
- Observing gravitational radiation from EMRIs will allow testing the black hole no-hair theorem
- It is crucial to compare the predictions from PN theory, perturbation theory and NR simulations:
  - Cross-check the validity of the various calculations
  - Determine domains of validity of approximation schemes