Black Hole Physics and Relativistic Celestial Mechanics

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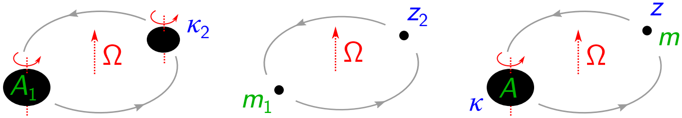


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THEMES

Black hole physics

General relativistic celestial mechanics

Gravitational-wave generation

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- General relativistic celestial mechanics
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METHODS

- Numerical relativity
- Post-Newtonian approximation
- Post-Minkowskian approximation
- Black hole perturbation theory
- Gravitational self-force theory
- Effective one-body model

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TOPICS

- Black hole tidal deformability
- Laws of compact binary mechanics
- Gauge-invariant comparisons

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Laws of compact binary mechanics

Gauge-invariant comparisons

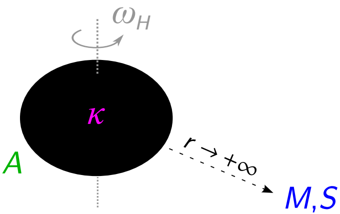
The black hole uniqueness theorem in GR

[Israel 1967, Carter 1971, Hawking 1973, Robinson 1975]

- The only stationary vacuum black hole solution is the Kerr solution of mass M and angular momentum S

"Black holes have no hair." (J. A. Wheeler)

- Black hole event horizon $\mathcal{H}$ characterized by:
  - Angular velocity ω_H
  - Surface gravity κ
  - Surface area A

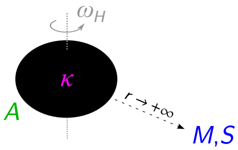


The laws of black hole mechanics

[Hawking 1972, Bardeen, Carter & Hawking 1973]

- Zeroth law of mechanics:

$$\kappa = \text{const.} \quad (\text{on } \mathcal{H})$$

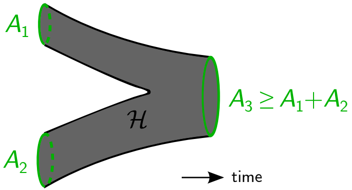


- First law of mechanics:

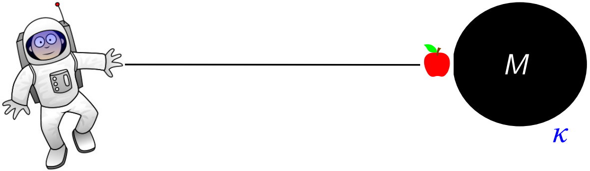
$$\delta M = \omega_H \, \delta S + \frac{\kappa}{8\pi} \, \delta A$$

- Second law of mechanics:

$$\delta A \geq 0$$



What is the horizon surface gravity?



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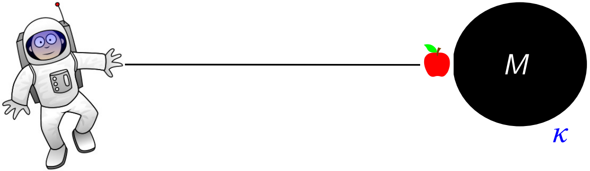

- For an event horizon $\mathcal{H}$ generated by a Killing field k^α:

$$\kappa^2 = \frac{1}{2} \left(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha \right) \Big|_{\mathcal{H}}$$

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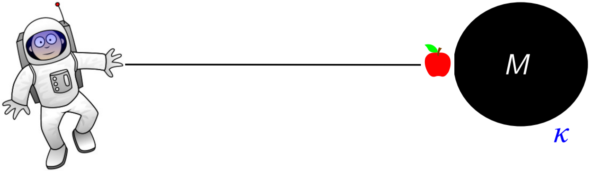

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$$\kappa^2 \equiv \frac{1}{2}\left(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha\right)\Big|_{\mathcal{H}}$$

- For a Schwarzschild black hole of mass M, this yields

$$\kappa = \frac{1}{4M} = \frac{GM}{R_S^2}$$

Beyond stationary, isolated black holes

Why?

- Astrophysical black holes are neither perfectly isolated, nor strictly stationary
- Of special interest are black holes that interact gravitationally with a companion in a compact binary system

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How?

- Isolated, slowly evolving, or dynamical horizons (quasi-local)
- Physical setup that guarantees the existence of an isometry
- Perturbative treatment of the problem: large separation, large mass ratio, weak tidal environment

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Outline

1. Circular-orbit binaries: geometrical methods
2. Beyond circular motion: Hamiltonian methods
3. Applications of the first law of binary mechanics

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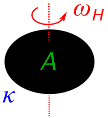
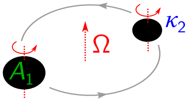
First laws of compact binary mechanics



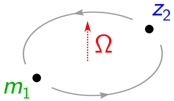
$$\delta M - \omega_H \, \delta S = \frac{\kappa}{8\pi} \, \delta A$$

[Bardeen *et al.* 1973]



$$\delta M - \Omega \, \delta J = \sum_a \frac{\kappa_a}{8\pi} \, \delta A_a$$

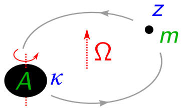
[Friedman *et al.* 2002]



$$\delta M - \Omega \, \delta J = \sum_a z_a \, \delta m_a$$

[Le Tiec *et al.* 2012]

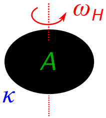
[Blanchet *et al.* 2013]



$$\delta M - \Omega \, \delta J = \frac{\kappa}{8\pi} \, \delta A + z \, \delta m$$

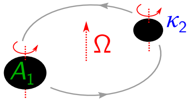
[Gralla & Le Tiec 2013]

First laws of compact binary mechanics



$$\delta M - \omega_H \, \delta S = 4 \mu \kappa \, \delta \mu$$

[Bardeen *et al.* 1973]



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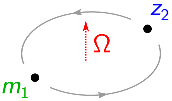

$$\delta M - \Omega \, \delta J = \sum_a z_a \, \delta m_a$$

[Le Tiec *et al.* 2012]

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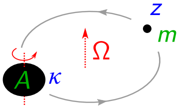
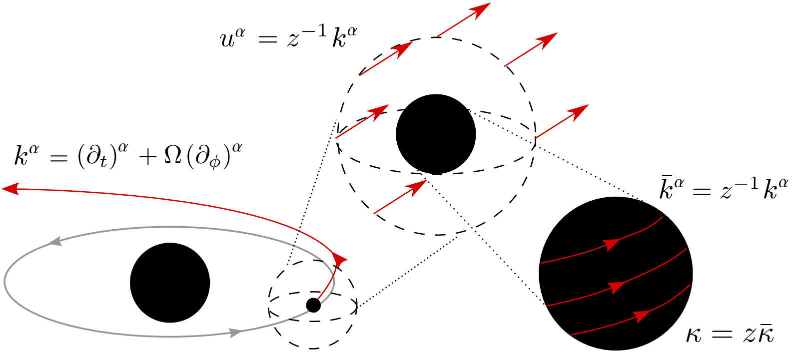


$$\delta M - \Omega \, \delta J = 4 \mu \kappa \, \delta \mu + z \, \delta m$$

[Gralla & Le Tiec 2013]

Surface gravity and redshift variable

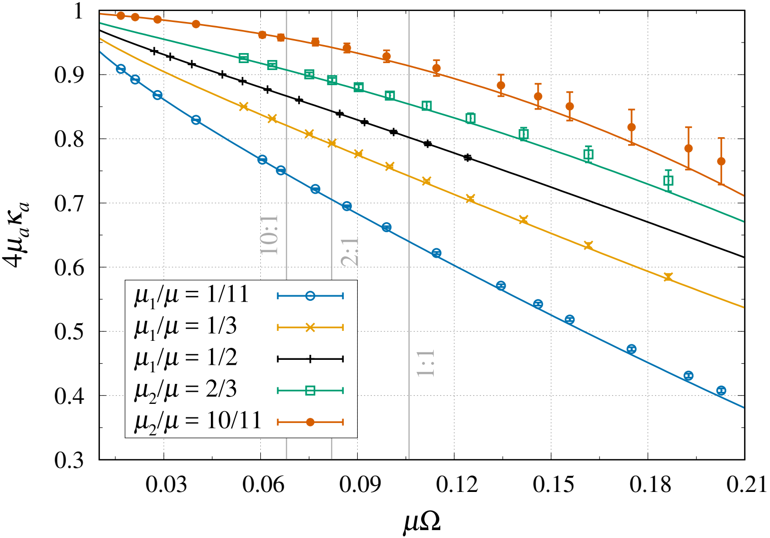
[Pound 2015]



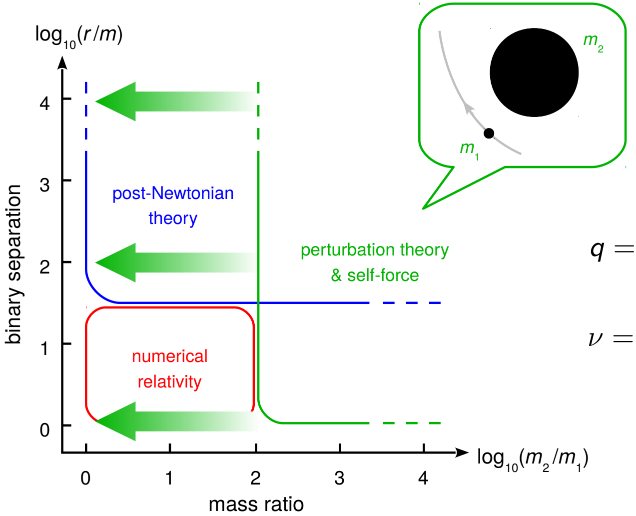
(Credit: Zimmerman, Lewis & Pfeiffer 2016)

Surface gravity *vs* orbital frequency

[Le Tiec & Grandclément 2018]



Perturbation theory for comparable masses



$$q = m_1/m_2$$
$$\downarrow$$
$$\nu = m_1 m_2/m^2$$

Perturbation theory for comparable masses

Comparisons to numerical relativity

conservative

- Periastron advance [Le Tiec *et al.* 2011, 2013]
- Binding energy [Le Tiec, Buonanno & Barausse 2012]
- Surface gravity [Zimmerman *et al.* 2016, Le Tiec & Grandclément 2018]

dissipative

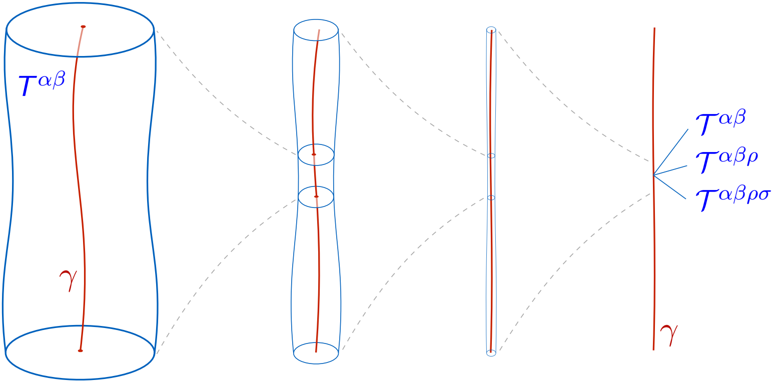
- Recoil velocity [Fitchett & Detweiler 1984, Nagar 2013]
- Head-on waveform [Anninos *et al.* 1995, Sperhake *et al.* 2011]
- Inspiral waveform [van de Meent & Pfeiffer 2020, Rifat *et al.* 2020]
- Inspiral energy flux [Warburton *et al.* 2021]

Structure of Einstein equation

- Polynomial nonlinearity using geometric variables [Harte 2014]
- Exact EOB energy map to $O(G)$ [Damour 2016, Vines 2017]

Multipolar gravitational skeleton

[Mathisson 1937, Tulczyjew 1957]



$$T^{\alpha\beta} \to T^{\alpha\beta}_{\text{skel}} = \int_\gamma \mathrm{d}\tau \left[\underbrace{\mathcal{T}^{\alpha\beta}\,\delta_4}_{\text{monopole}} + \underbrace{\nabla_\rho \left(\mathcal{T}^{\alpha\beta\rho}\,\delta_4\right)}_{\text{dipole}} + \cdots \right]$$

Multipolar gravitational skeleton

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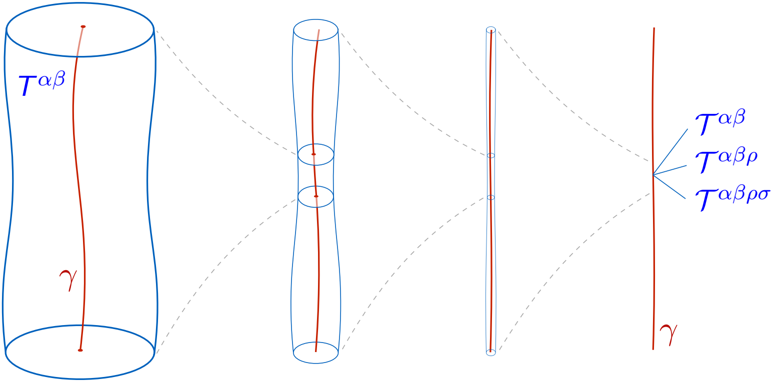


$$T^{\alpha\beta} \to T^{\alpha\beta}_{\text{skel}} = \int_\gamma \mathrm{d}\tau \left[\underbrace{u^{(\alpha}p^{\beta)}\,\delta_4}_{\text{monopole}} + \underbrace{\nabla_\rho\left(u^{(\alpha}S^{\beta)\rho}\,\delta_4\right)}_{\text{dipole}} + \cdots \right]$$

Quadrupolar particles on a circular orbit

[Ramond & Le Tiec 2021a]

- Helical Killing field k^α so that

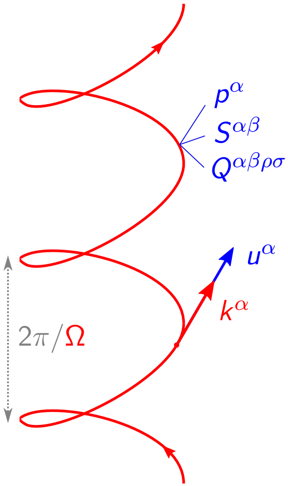
$$\mathcal{L}_k g_{\alpha\beta} = 0$$

- Each particle worldline γ is an integral curve of k^α:

$$k^\alpha|_\gamma = z\, u^\alpha$$

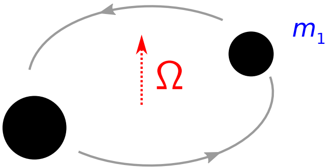
- The particle multipoles are all Lie-dragged along k^α:

$$\mathcal{L}_k p^\alpha = \mathcal{L}_k S^{\alpha\beta} = \mathcal{L}_k Q^{\alpha\beta\rho\sigma} = 0$$



First law with leading finite-size effects

[Ramond & Le Tiec 2021b]



$$\delta M - \Omega\,\delta J = \sum_a |k|_a\,\delta m_a$$

First law with leading finite-size effects

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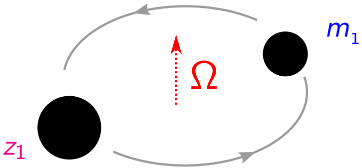

$$\delta M - \Omega\,\delta J = \sum_a z_a\,\delta m_a$$

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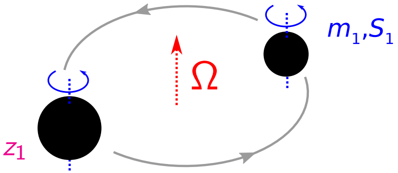
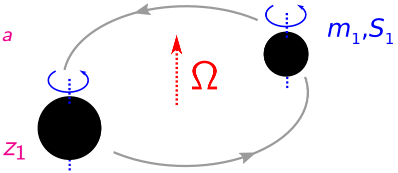


$$\delta M - \Omega\, \delta J = \sum_a z_a\, \delta m_a - \sum_a |\nabla k|_a\, \delta S_a$$

First law with leading finite-size effects

[Ramond & Le Tiec 2021b]

$\dot{\mathbf{S}}_a = \boldsymbol{\Omega}_a \times \mathbf{S}_a$



$$\delta M - \Omega \, \delta J = \sum_a z_a \, \delta m_a - \sum_a (\Omega - \Omega_a) \, \delta S_a$$

First law with leading finite-size effects

[Ramond & Le Tiec 2021c]

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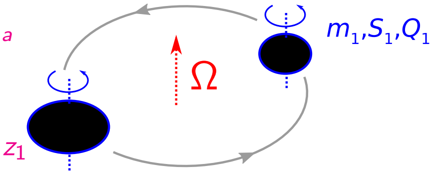

$$\delta M - \Omega\,\delta J = \sum_a z_a\,\delta m_a - \sum_a (\Omega - \Omega_a)\,\delta S_a + \sum_a |\nabla\nabla k|_a\,\delta Q_a$$

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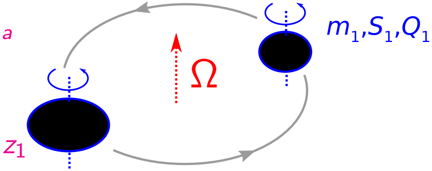

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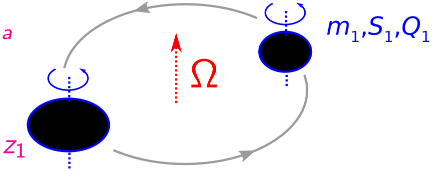

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- Spin-induced quadrupole $Q_{\text{spin}} \sim \kappa S^2$
- Tidally-induced quadrupole $Q_{\text{tidal}} \sim \lambda \mathcal{E}$

Tidal deformability of Kerr black holes

[Le Tiec, Casals & Franzin 2021]

$$Q_{ij}^{\text{spin}} = -S_{\langle i}S_{j\rangle}/M \quad \text{and} \quad Q_{ij}^{\text{tidal}} = \frac{16}{45}M^3\mathcal{E}^k{}_{(i}\epsilon_{j)kl}S^l$$

Consistent with known tidal torquing of spinning black holes

Outline

1. Circular-orbit binaries: geometrical methods
2. Beyond circular motion: Hamiltonian methods
3. Applications of the first law of binary mechanics

Averaged redshift for eccentric orbits

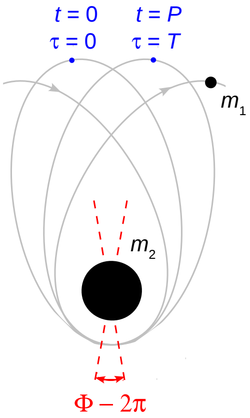
[Barack & Sago 2011]

- Generic eccentric orbit parameterized by the two requencies

$$\Omega_r = \frac{2\pi}{P}\,, \quad \Omega_\phi = \frac{\Phi}{P}$$

- Time average of redshift $z = \mathrm{d}\tau/\mathrm{d}t$ over one radial period

$$\langle z \rangle \equiv \frac{1}{P}\int_0^P z(t)\,\mathrm{d}t = \frac{1}{P}\int_0^{\mathcal{T}} \mathrm{d}\tau = \frac{\mathcal{T}}{P}$$



First law of mechanics for eccentric orbits

[Le Tiec 2015, Blanchet & Le Tiec 2017]

- Canonical ADM Hamiltonian $H(\mathbf{x}_a, \mathbf{p}_a; m_a)$ of two point particles with constant masses m_a
- Variation δH + Hamilton's equation + orbital averaging:

$$\delta M = \Omega_\phi \, \delta L + \Omega_r \, \delta J_r + \sum_a \langle z_a \rangle \, \delta m_a$$

- Starting at 4PN order the binary dynamics gets nonlocal in time because of gravitational-wave tails:

$$H_{\text{tail}}^{\text{4PN}}[\mathbf{x}_a(t), \mathbf{p}_a(t)] = -\frac{M}{5} I_{ij}^{(3)}(t) \underset{2r}{\text{Pf}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{\tau} \, I_{ij}^{(3)}(t+\tau)$$

- With appropriate M, L and J_r the first law still holds

Particle Hamiltonian first law

- Geodesic motion of test mass m in Kerr geometry $\bar{g}_{\alpha\beta}$ derives from Hamiltonian

$$\bar{H}(x,p) = \frac{1}{2}\,\bar{g}^{\alpha\beta}(x^\mu)p_\alpha p_\beta$$

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- Hamilton-Jacobi equation is completely separable [Carter 1968]
- Canonical transformation $(x^\mu, p_\mu) \to (q^\alpha, J_\alpha)$ to *generalized* action-angle variables [Schmidt 2002, Hinderer & Flanagan 2008]

$$\frac{\mathrm{d}J_\alpha}{\mathrm{d}\tau} = -\frac{\partial \bar{H}}{\partial q^\alpha} = 0\,, \quad \frac{\mathrm{d}q^\alpha}{\mathrm{d}\tau} = \frac{\partial \bar{H}}{\partial J_\alpha} \equiv \omega^\alpha$$

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- Varying $\bar{H}(J_\alpha)$ yields a particle Hamiltonian first law valid for *generic* bound orbits [Le Tiec 2014]

$$\delta E = \Omega_\phi\,\delta L + \Omega_r\,\delta J_r + \Omega_\theta\,\delta J_\theta + \langle z\rangle\,\delta m$$

Including conservative self-force effects

[Fujita, Isoyama, Le Tiec, Nakano, Sago & Tanaka 2017]

- Geodesic motion of self-gravitating mass m in effective metric $\bar{g}_{\alpha\beta} + h^{\mathrm{R}}_{\alpha\beta}$ derives from Hamiltonian

$$H(x, p; \gamma) = \bar{H}(x, p) + H_{\mathrm{int}}(x, p; \gamma)$$

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- In class of canonical gauges, one can define a *unique* effective Hamiltonian $\mathcal{H}(J) = \bar{H}(J) + \frac{1}{2}\langle H_{\mathrm{int}}\rangle(J)$ yielding a first law valid for *generic* bound orbits:

$$\delta\mathcal{E} = \Omega_\phi\, \delta\mathcal{L} + \Omega_r\, \delta\mathcal{J}_r + \Omega_\theta\, \delta\mathcal{J}_\theta + \langle z\rangle\, \delta m$$

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$$\delta\mathcal{E} = \Omega_\phi\,\delta\mathcal{L} + \Omega_r\,\delta\mathcal{J}_r + \Omega_\theta\,\delta\mathcal{J}_\theta + \langle z\rangle\,\delta m$$

- The actions $\mathcal{J}_\alpha$ and the averaged redshift $\langle z\rangle$, as functions of $(\Omega_r,\Omega_\theta,\Omega_\phi)$, include conservative self-force corrections from the *gauge-invariant* averaged interaction Hamiltonian $\langle H_{\mathrm{int}}\rangle$

Outline

Applications of the first laws

- Fix 'ambiguity parameters' in 4PN two-body equations of motion
  [Jaranowski & Schäfer 2012, Damour *et al.* 2014, Bernard *et al.* 2016]
- Inform the 5PN two-body Hamiltonian in a 'tutti-frutti' method
  [Bini, Damour & Geralico 2019, 2020]
- Compute GSF contributions to energy and angular momentum
  [Le Tiec, Barausse & Buonanno 2012]
- Calculate Schwarzschild and Kerr ISCO frequency shifts
  [Le Tiec *et al.* 2012, Akcay *et al.* 2012, Isoyama *et al.* 2014]
- Test cosmic censorship conjecture including GSF effects
  [Colleoni & Barack 2015, Colleoni *et al.* 2015]
- Calibrate EOB potentials in effective Hamiltonian
  [Barausse *et al.* 2012, Akcay & van de Meent 2016, Bini *et al.* 2016]
- Compare particle redshift to black hole surface gravity
  [Zimmerman, Lewis & Pfeiffer 2016, Le Tiec & Grandclément 2018]
- Benchmark for calculations of Schwarzschild IBCO frequency shift and gravitational binding energy [Barack *et al.* 2019, Pound *et al.* 2020]

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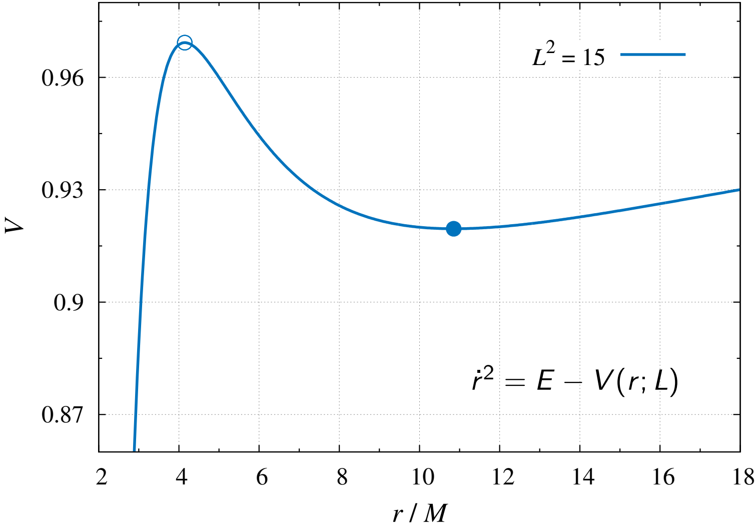
  [Le Tiec, Barausse & Buonanno 2012]
- ✔ Calculate Schwarzschild and Kerr ISCO frequency shifts
  [Le Tiec *et al.* 2012, Akcay *et al.* 2012, Isoyama *et al.* 2014]
- Test cosmic censorship conjecture including GSF effects
  [Colleoni & Barack 2015, Colleoni *et al.* 2015]
- Calibrate EOB potentials in effective Hamiltonian
  [Barausse *et al.* 2012, Akcay & van de Meent 2016, Bini *et al.* 2016]
- Compare particle redshift to black hole surface gravity
  [Zimmerman, Lewis & Pfeiffer 2016, Le Tiec & Grandclément 2018]
- ✔ Benchmark for calculations of Schwarzschild IBCO frequency shift and gravitational binding energy [Barack *et al.* 2019, Pound *et al.* 2020]

Innermost stable circular orbit (ISCO)



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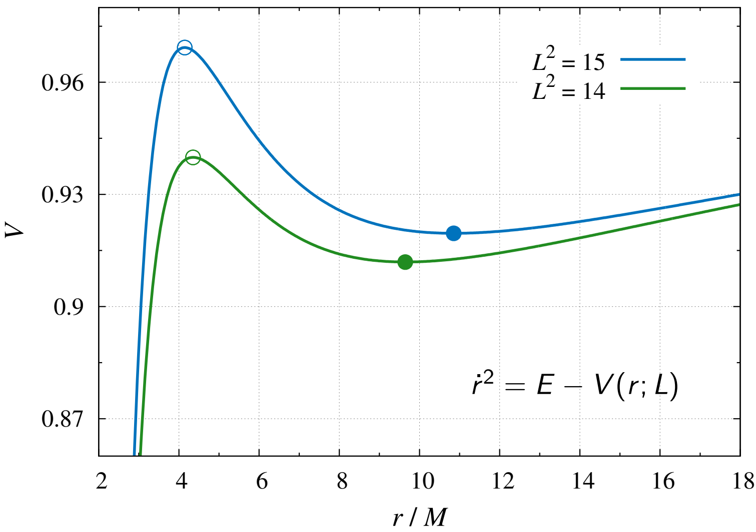

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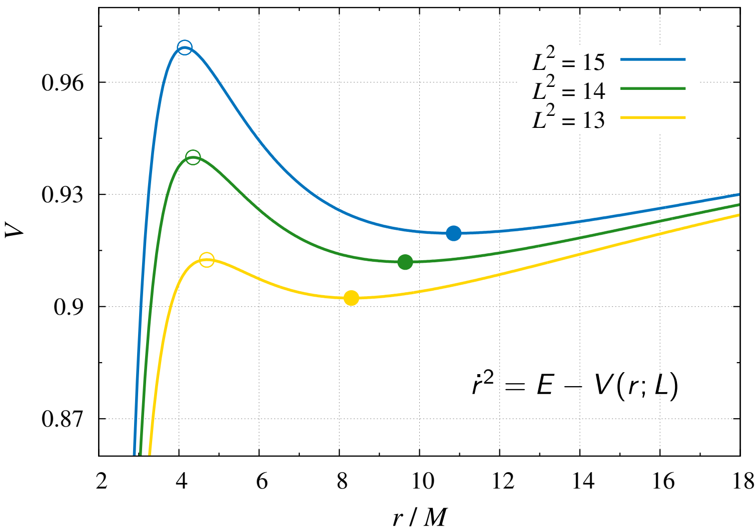

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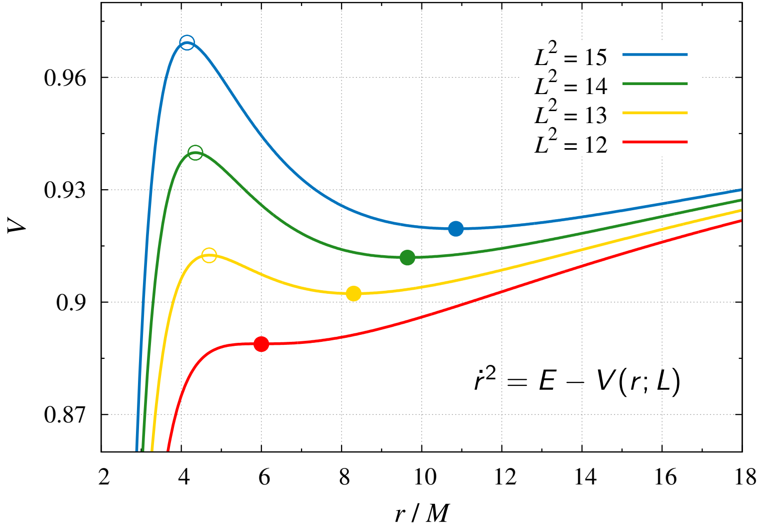

Frequency shift of the Kerr ISCO

[Isoyama *et al.* 2014]

- The orbital frequency of the Kerr ISCO is shifted under the effect of the conservative self-force:

$$(M+\mu)\Omega_{\text{isco}} = \underbrace{M\Omega^{(0)}_{\text{isco}}(\chi)}_{\substack{\text{test mass}\\ \text{result}}} \left[1 + \underbrace{q\, C_\Omega(\chi)}_{\substack{\text{self-force}\\ \text{correction}}} + O(q^2)\right]$$

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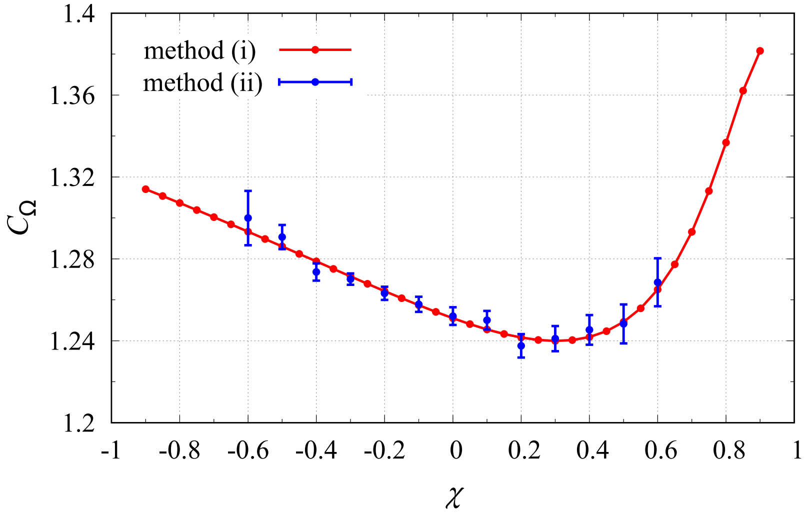
- The frequency shift can be computed from a stability analysis of slightly eccentric orbits near the Kerr ISCO

- Combining the Hamiltonian first law with the condition $\partial E/\partial\Omega = 0$ yields the same result:

$$C_\Omega = 1 + \frac{1}{2}\,\frac{z''_{(1)}(\Omega^{(0)}_{\text{isco}})}{\hat{E}''_{(0)}(\Omega^{(0)}_{\text{isco}})}$$

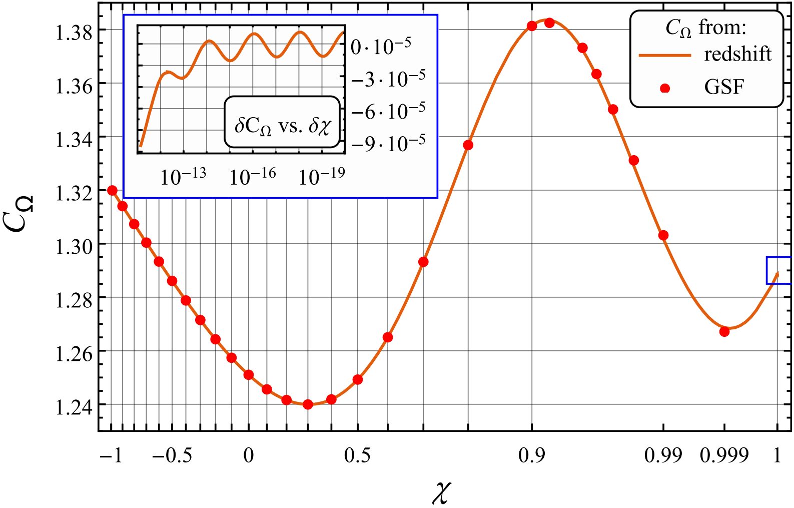
ISCO frequency shift *vs* black hole spin

[Isoyama *et al.* 2014]



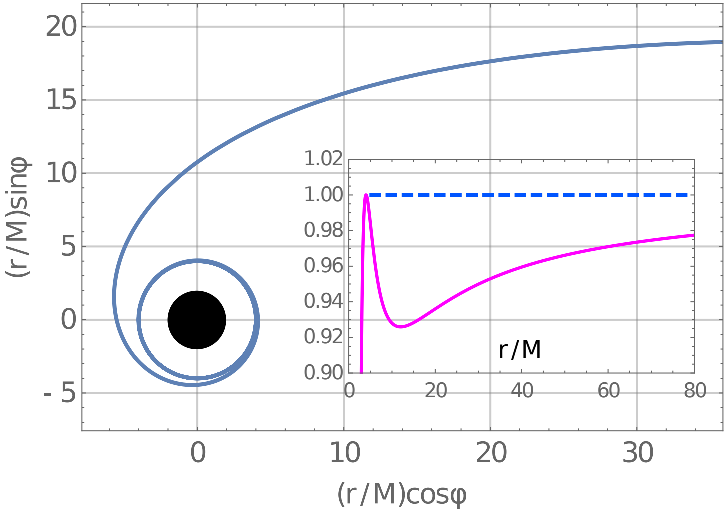
ISCO frequency shift *vs* black hole spin

[van de Meent 2017]



Innermost bound orbit in Schwarzschild

[Barack, Colleoni, Damour, Isoyama & Sago 2019]



IBCO frequency shift in Schwarzschild

[Barack, Colleoni, Damour, Isoyama & Sago 2019]

Direct GSF calculation

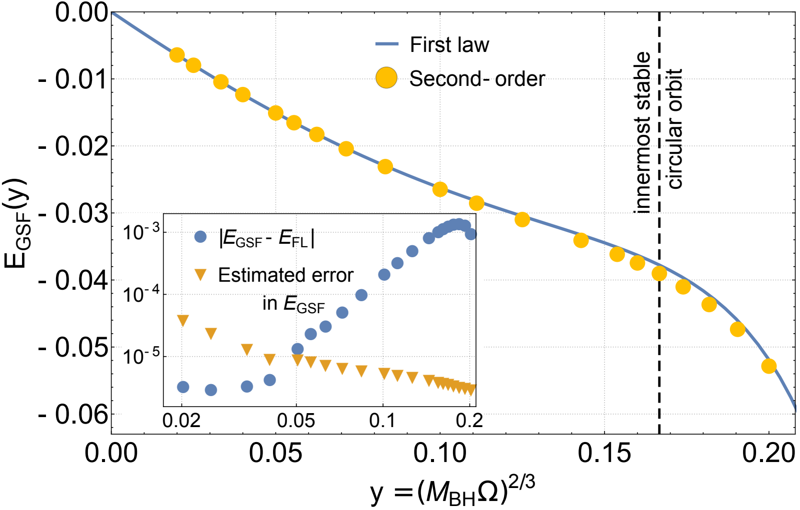
$$\Omega_{\mathrm{IBCO}} = (8M)^{-1}\left[1 + 0.5536(2)\,q + O(q^2)\right]$$
$$J_{\mathrm{IBCO}} = 4M\mu\left[1 - 0.304(3)\,q + O(q^2)\right]$$

First-law prediction

$$\Omega_{\mathrm{IBCO}} = (8M)^{-1}\left[1 + 0.55360302918(2)\,q + O(q^2)\right]$$
$$J_{\mathrm{IBCO}} = 4M\mu\left[1 - 0.304674287863142(6)\,q + O(q^2)\right]$$

Binding energy *vs* orbital frequency

[Pound, Wardell, Warburton & Miller 2020]



Summary

- The classical laws of black hole mechanics can be extended to binary systems of compact objects
- First laws of mechanics come in a variety of different forms:
  - Context: exact GR, self-force theory, PN theory
  - Objects: black holes, multipolar point particles
  - Orbits: corotating, circular, eccentric, generic
  - Derivation: geometric, Hamiltonian
- Combined with the first law, the redshift $z(\Omega)$ provides crucial information about the binary dynamics:
  - Gravitational binding energy E and angular momentum J
  - ISCO frequency Ω_{ISCO} and IBCO frequency Ω_{IBCO}
  - EOB effective potentials A, $\bar{D}$, Q, ...
  - Horizon surface gravity κ

Prospects

- Small-mass-ratio approximation useful to build templates for IMRIs and even comparable-mass binaries
- Exploit the Hamiltonian first law for a particle in Kerr:
    - Innermost spherical orbits
    - Unbound zoom-whirl orbits
- Extend Hamiltonian first law for two spinning particles:
    - Non-aligned spins and generic precessing orbits
    - Contribution from quadrupole moments
- Link to unbound orbits and scattering angle via analytic continuation?
- Derive a first law in post-Minkowskian gravity
- Derive a first law with dissipation

Additional Material

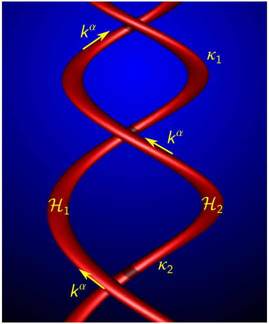
Generalized zeroth law of mechanics

[Friedman, Uryū & Shibata 2002]

- Black hole spacetimes with *helical* Killing vector field k^α
- On each component $\mathcal{H}_i$ of the event horizon, the expansion and shear of the generators vanish
- Generalized rigidity theorem: $\mathcal{H} = \bigcup_i \mathcal{H}_i$ is a Killing horizon
- *Constant* horizon surface gravity

$$\kappa_i^2 = \frac{1}{2}(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha)\Big|_{\mathcal{H}_i}$$

- The binary black hole system is in a state of *corotation*

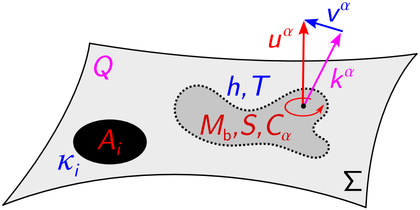


Generalized first law of mechanics

[Friedman, Uryū & Shibata 2002]

- Spacetimes with black holes + perfect fluid matter sources
- One-parameter family of solutions $\{g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)\}$
- *Globally* defined Killing field $k^\alpha \rightarrow$ conseved Noether charge Q

$$\delta Q = \sum_i \frac{\kappa_i}{8\pi}\,\delta A_i + \int_\Sigma \left[\, \bar{h}\,\delta(\mathrm{d}M_\mathrm{b}) + \bar{T}\delta(\mathrm{d}S) + v^\alpha\delta(\mathrm{d}C_\alpha)\right]$$



Issue of asymptotic flatness

[Friedman, Uryū & Shibata 2002]

- Binaries on circular orbits have a *helical* Killing symmetry k^α
- Helically symmetric spacetimes are *not* asymptotically flat
  [Gibbons & Stewart 1983, Detweiler 1989, Klein 2004]
- Asymptotic flatness can be recovered if radiation (reaction) can be "turned off":
  - Conformal Flatness Condition
  - Post-Newtonian approximation
  - Black hole perturbation theory
- For asymptotically flat spacetimes:

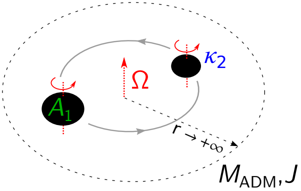
$$k^\alpha \to t^\alpha + \Omega\,\phi^\alpha \quad \text{and} \quad \delta Q = \delta M_{\mathrm{ADM}} - \Omega\,\delta J$$

Application to black hole binaries

[Friedman, Uryū & Shibata 2002]

- Rigidity theorem $\rightarrow$ black holes are in a state of corotation
- Conformal flatness condition $\rightarrow$ asymptotic flatness recovered
  ↳ preferred normalization of κ_i [Le Tiec & Grandclément 2018]
- For binary black holes the generalized first law reduces to

$$\delta M_{\mathrm{ADM}} = \Omega\,\delta J + \sum_i \frac{\kappa_i}{8\pi}\,\delta A_i$$



First law for point-particle binaries

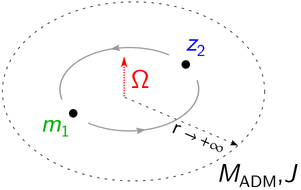
[Le Tiec, Blanchet & Whiting 2012]

- For balls of dust, the generalized first law reduces to

$$\delta Q = \int_\Sigma z\, \delta(\mathrm{d}M_\mathrm{b}) + \cdots\,, \quad \text{where} \quad z = -k^\alpha u_\alpha$$

- Conservative PN dynamics $\rightarrow$ asymptotic flatness recovered
- Two *spinless* compact objects modelled as point masses m_i and moving along circular orbits obey the first law

$$\delta M_\mathrm{ADM} = \Omega\, \delta J + \sum_i z_i\, \delta m_i$$



Extension to spinning binaries

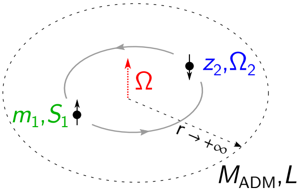
[Blanchet, Buonanno & Le Tiec 2013]

- Canonical ADM Hamiltonian $H(\mathbf{x}_i, \mathbf{p}_i, \mathbf{S}_i; m_i)$ of two point particles with masses m_i and spins $\mathbf{S}_i$ [Steinhoff *et al.* 2008]
- Redshift observables and spin precession frequencies:

$$\frac{\partial H}{\partial m_i} = z_i \quad \text{and} \quad \frac{\partial H}{\partial \mathbf{S}_i} = \mathbf{\Omega}_i$$

- First law for aligned spins ($J = L + \sum_i S_i$) and circular orbits:

$$\delta M = \Omega\, \delta L + \sum_i \left(z_i\, \delta m_i + \Omega_i\, \delta S_i\right)$$



Corotating point particles

[Blanchet, Buonanno & Le Tiec 2013]

- A point particle with rest mass m_i and spin S_i is given an *irreducible* mass μ_i and a proper rotation frequency ω_i *via*

$$\delta m_i = \omega_i \, \delta S_i + c_i \, \delta \mu_i \quad \text{and} \quad m_i^2 = \mu_i^2 + S_i^2/(4\mu_i^2)$$

- The first law of binary point-particle mechanics becomes

$$\delta M = \Omega \, \delta J + \sum_i \left[z_i c_i \, \delta \mu_i + (z_i \, \omega_i + \Omega_i - \Omega) \, \delta S_i \right]$$

- Comparing with the first law for *corotating* black holes, $\delta M = \Omega \, \delta J + \sum_i (4\mu_i \kappa_i) \, \delta \mu_i$, the corotation condition is

$$z_i \, \omega_i = \Omega - \Omega_i \quad \longrightarrow \quad \omega_i(\Omega) \quad \longrightarrow \quad S_i(\Omega)$$

Surface gravity and redshift

[Blanchet, Buonanno & Le Tiec 2013]

- First law for corotating black holes

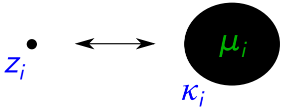
$$\delta M = \Omega\,\delta J + \sum_i (4\mu_i\kappa_i)\,\delta\mu_i$$

- First law for corotating point particles

$$\delta M = \Omega\,\delta J + \sum_i z_i c_i\,\delta\mu_i$$

- Analogy between BH surface gravity and particle redshift

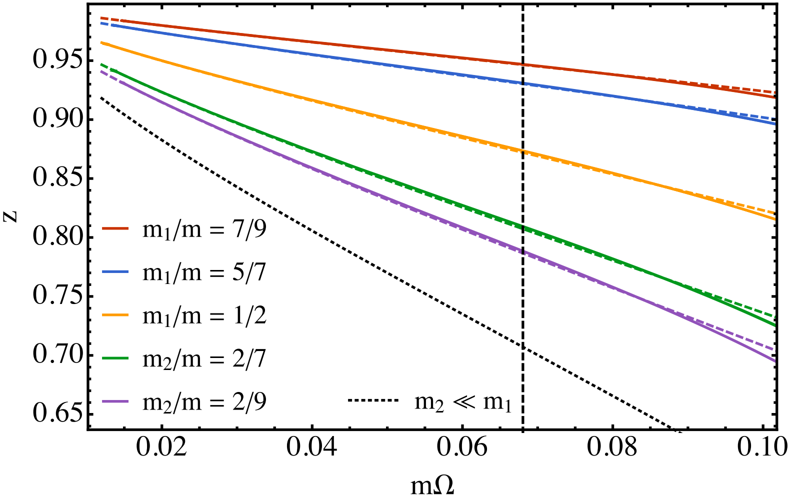
$$4\mu_i\kappa_i \longleftrightarrow z_i c_i$$



- New *invariant* relations for NR/BHP/PN comparison: $\kappa_i(\Omega)$

Redshift *vs* orbital frequency

[Zimmerman, Lewis & Pfeiffer 2016]



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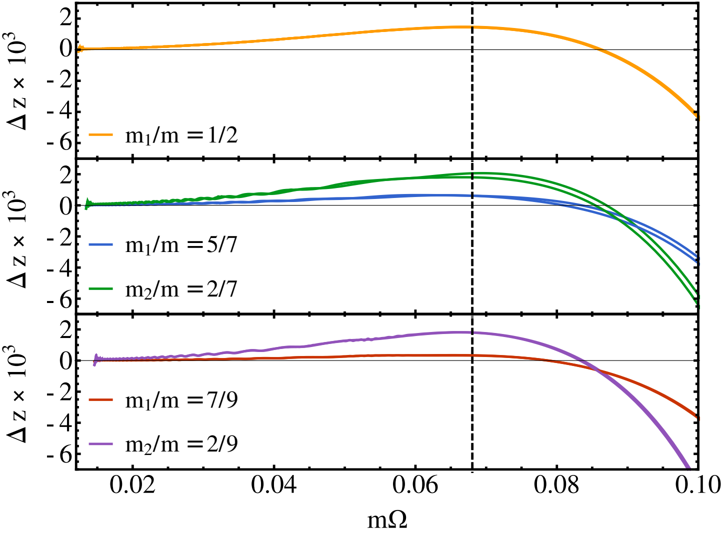


Quasi-equilibrium initial data

- 3+1 decomposition of the metric:

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \gamma_{ij} \left(\mathrm{d}x^i + N^i \mathrm{d}t\right) \left(\mathrm{d}x^j + N^j \mathrm{d}t\right)$$

- Conformal flatness condition approximation:

$$\gamma_{ij} = \Psi^4 f_{ij} + \cancel{h_{ij}}$$

- Assume *exact* helical Killing symmetry:

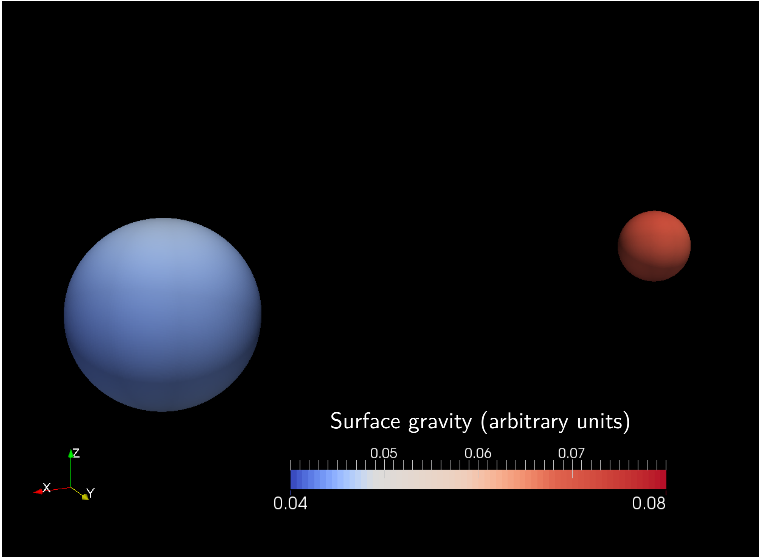
$$\mathcal{L}_k \, g_{\alpha\beta} = 0 \quad \text{with} \quad k^\alpha = (\partial_t)^\alpha + \Omega \, (\partial_\phi)^\alpha$$

- Solve five elliptic equations for (N, N^i, Ψ)
- Determine orbital frequency Ω by imposing

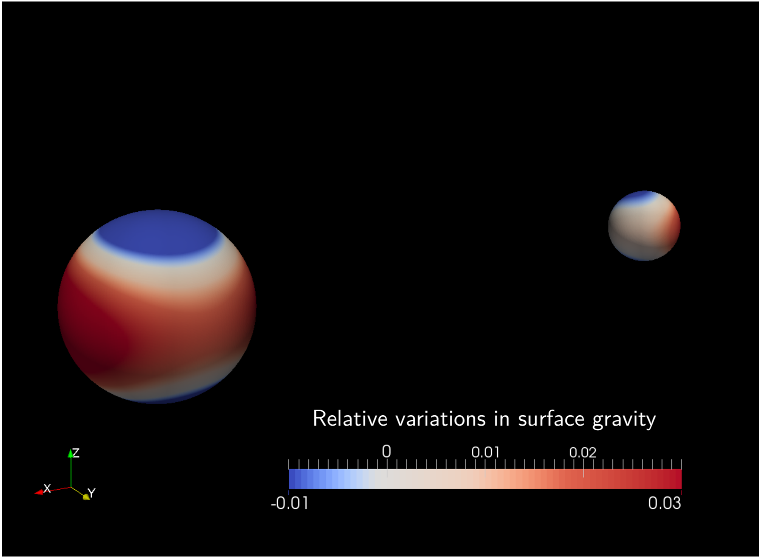
$$M_{\mathrm{ADM}} = M_{\mathrm{Komar}}$$

- Impose vanishing linear momentum to find rotation axis

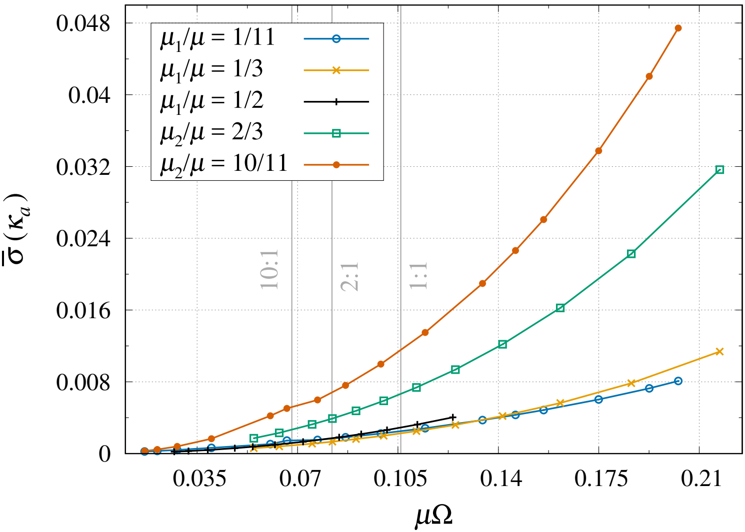
Surface gravity for mass ratio 2 : 1



Surface gravity for mass ratio 2 : 1



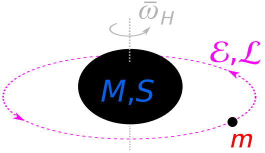
Variations in horizon surface gravity



Rotating black hole + orbiting moon

- Kerr black hole of mass M and spin S perturbed by a moon of mass $m \ll M$:

$$g_{ab}(\varepsilon) = \bar{g}_{ab} + \varepsilon\, \mathcal{D} g_{ab} + \mathcal{O}(\varepsilon^2)$$



- Perturbation $\mathcal{D} g_{ab}$ obeys the linearized Einstein equation with point-particle source

$$\mathcal{D} G_{ab} = 8\pi\, \mathcal{D} T_{ab} = 8\pi\, m \int_\gamma \mathrm{d}\tau\, \delta_4(x, y)\, u_a u_b$$

- Particle has energy $\mathcal{E} = -m\, t^a u_a$ and ang. mom. $\mathcal{L} = m\, \phi^a u_a$
- Physical $\mathcal{D} g_{ab}$: retarded solution, no incoming radiation, perturbations $\mathcal{D} M_{\mathrm{B}} = \mathcal{E}$ and $\mathcal{D} J = \mathcal{L}$ [Keidl *et al.* 2010]

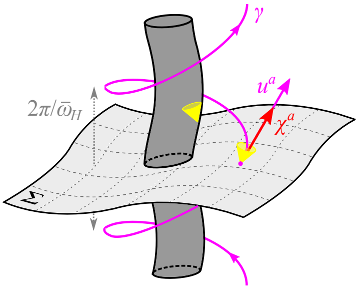
Rotating black hole + corotating moon

- We choose for the geodesic γ the unique equatorial, circular orbit with azimuthal frequency $\bar{\omega}_H$, i.e., the *corotating* orbit
- Gravitational radiation-reaction is $\mathcal{O}(\varepsilon^2)$ and neglected
  The spacetime geometry has a helical symmetry
- In adapted coordinates, the helical Killing field reads

$$\chi^a = t^a + \bar{\omega}_H \, \phi^a$$

- Conserved orbital quantity associated with symmetry:

$$z \equiv -\chi^a u_a = m^{-1} \, (\mathcal{E} - \bar{\omega}_H \, \mathcal{L})$$



Zeroth law for a black hole with moon

[Gralla & Le Tiec 2013]

- Because of helical symmetry and corotation, the expansion and shear of the *perturbed* future event horizon H vanish
- Rigidity theorems then imply that H is a Killing horizon
  [Hawking 1972, Chruściel 1997, Friedrich *et al.* 1999, etc]
- The horizon-generating Killing field must be of the form

$$k^a(\varepsilon) = t^a + \big(\underbrace{\bar{\omega}_H + \varepsilon\,\mathcal{D}\omega_H}_{\substack{\text{circular orbit}\\ \text{frequency } \Omega}}\big)\phi^a + \mathcal{O}(\varepsilon^2)$$

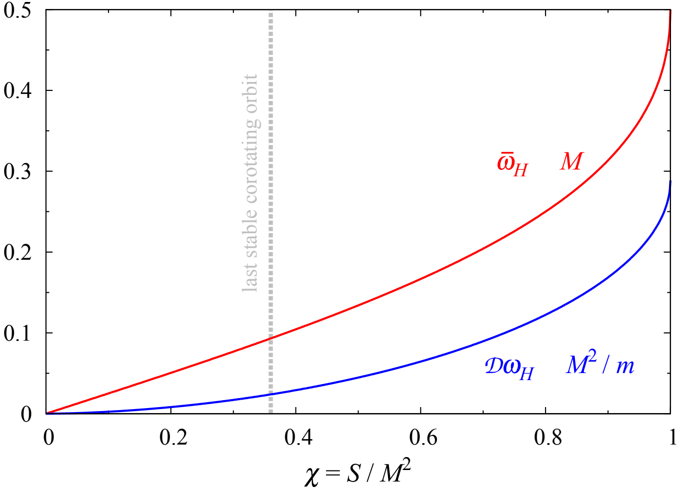
- The surface gravity κ is defined in the usual manner as

$$\kappa^2 = -\frac{1}{2}\,(\nabla^a k^b\,\nabla_a k_b)|_H$$

- Since $\kappa = \text{const.}$ over *any* Killing horizon [Bardeen *et al.* 1973], we have proven a zeroth law for the *perturbed* event horizon

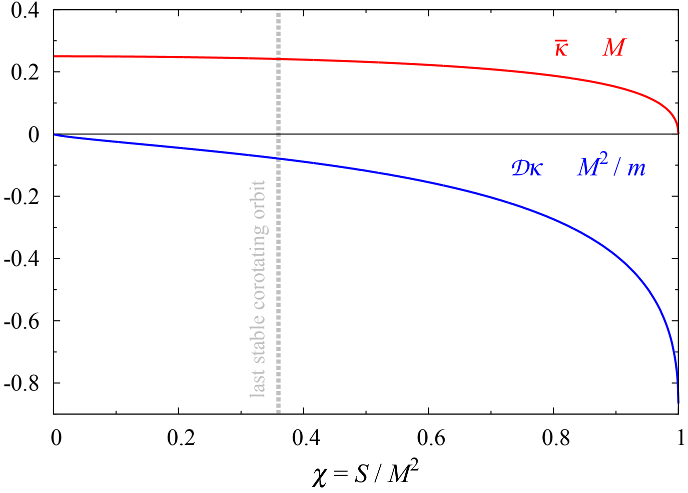
Angular velocity *vs* black hole spin

[Gralla & Le Tiec 2013]



Surface gravity *vs* black hole spin

[Gralla & Le Tiec 2013]



First law for a black hole with moon

[Gralla & Le Tiec 2013]

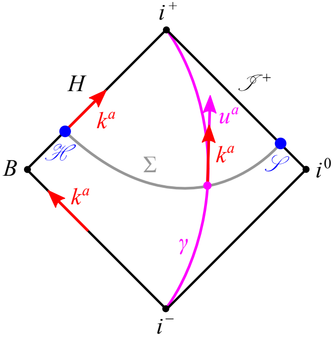
- Adapting [Iyer & Wald 1994] to non-vacuum perturbations of non-stationary spacetimes we find (with $Q_{ab} \equiv -\varepsilon_{abcd} \nabla^c k^d$)

$$\int_{\partial\Sigma} (\delta Q_{ab} - \Theta_{abc} k^c) = 2\,\delta \int_{\Sigma} \varepsilon_{abcd} G^{de} k_e - \int_{\Sigma} \varepsilon_{abcd} k^d G^{ef} \delta g_{ef}$$

- Applied to nearby BH with moon spacetimes, this gives the first law

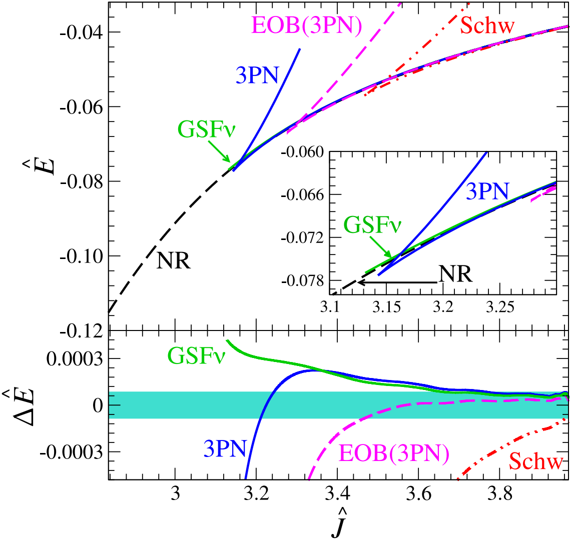
$$\delta M_{\mathrm{B}} = \Omega\,\delta J + \frac{\kappa}{8\pi}\,\delta A + z\,\delta m$$

- Features variations of the Bondi mass and angular momentum



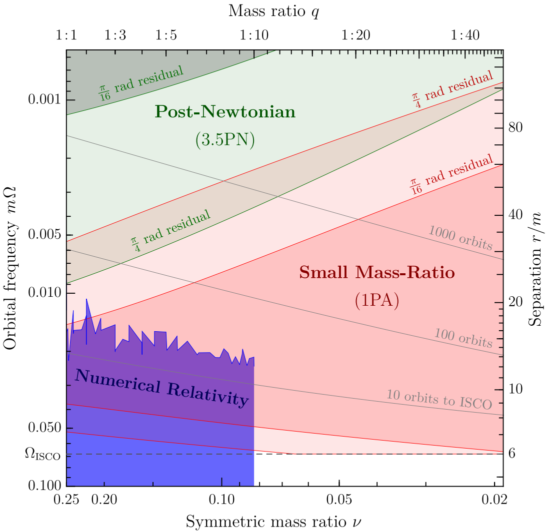
Binding energy *vs* angular momentum

[Le Tiec, Barausse & Buonanno 2012]



Perturbation theory for comparable masses

[van de Meent & Pfeiffer 2020]



$$q = m_1/m_2$$
$$\downarrow$$
$$\nu = m_1 m_2/m^2$$

Why does BHPT perform so well?

- In perturbation theory, one traditionally expands as

$$f(\Omega; m_i) = \sum_{k=0}^{k_{\max}} a_k(m_2\Omega)\, q^k \quad \text{where} \quad q \equiv m_1/m_2 \in [0,1]$$

- However, most physically interesting relationships $f(\Omega; m_i)$ are symmetric under exchange $m_1 \longleftrightarrow m_2$
- Hence, a better-motivated expansion is

$$f(\Omega; m_i) = \sum_{k=0}^{k_{\max}} b_k(m\Omega)\, \nu^k \quad \text{where} \quad \nu \equiv m_1 m_2/m^2 \in [0,1/4]$$

- In a PN expansion, we have $b_n = \mathcal{O}\big(1/c^{2n}\big) = n\text{PN} + \cdots$

Why does BHPT perform so well?

- In perturbation theory, each surface gravity is expanded as

$$4\mu_1\kappa_1 = a(\mu_2\Omega) + q\, b(\mu_2\Omega) + \mathcal{O}(q^2)$$
$$4\mu_2\kappa_2 = c(\mu_2\Omega) + q\, d(\mu_2\Omega) + \mathcal{O}(q^2)$$

- From the first law we know that the general form is

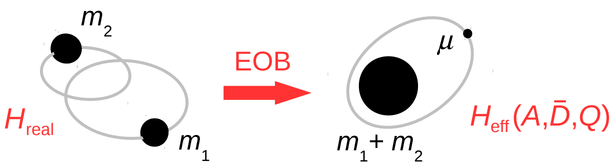
$$4\mu_i\kappa_i = \sum_{k\geqslant 0} \nu^k f_k(\mu\Omega) \pm \sqrt{1-4\nu} \sum_{k\geqslant 0} \nu^k g_k(\mu\Omega)$$

- Each surface gravity can thus be rewritten as

$$\begin{aligned} 4\mu_i\kappa_i = {} & A(\mu\Omega) \pm B(\mu\Omega)\sqrt{1-4\nu} + C(\mu\Omega)\,\nu \\ & \pm D(\mu\Omega)\,\nu\sqrt{1-4\nu} + \mathcal{O}(\nu^2) \end{aligned}$$

- Expand to linear order in q and match $\rightarrow A, B, C, D$

EOB dynamics beyond circular motion



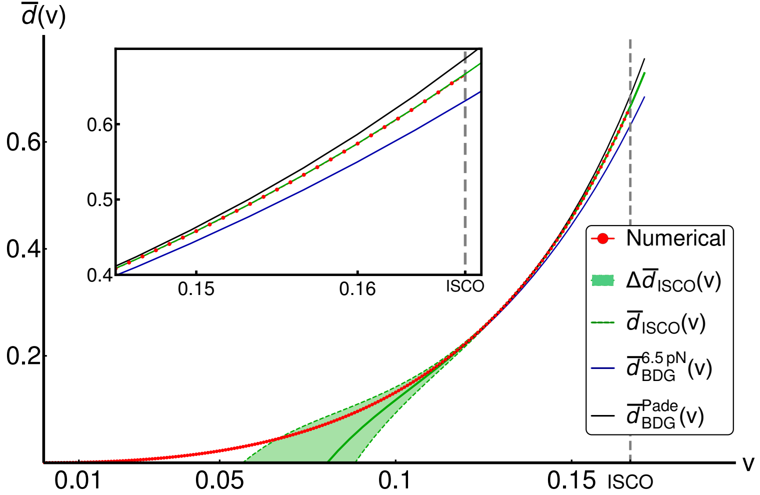
- Conservative EOB dynamics determined by "potentials"

$$A(r) = 1 - 2M/r + \nu\, a(r) + \cdots$$
$$\bar{D}(r) = 1 + \nu\, \bar{d}(r) + \cdots$$
$$Q(r) = \nu\, q(r)\, p_r^4 + \cdots$$

- Functions $a(r)$, $\bar{d}(r)$ and $q(r)$ controlled by $\langle z \rangle_{\text{GSF}}(\Omega_r, \Omega_\phi)$

EOB dynamics beyond circular motion

[Akcay & van de Meent 2016]



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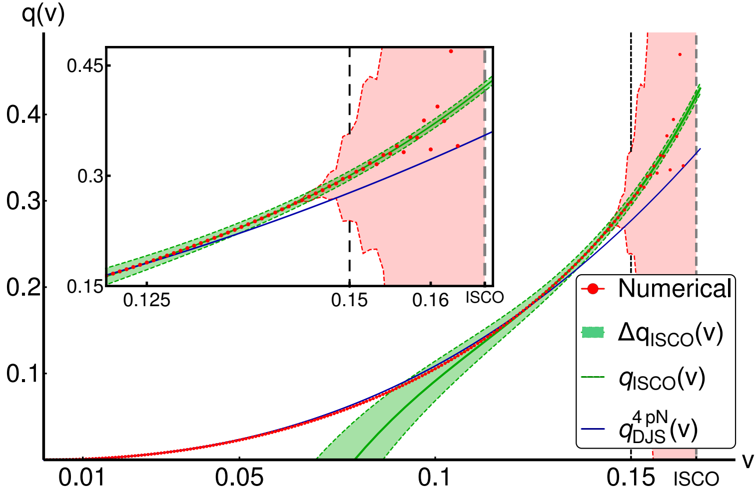
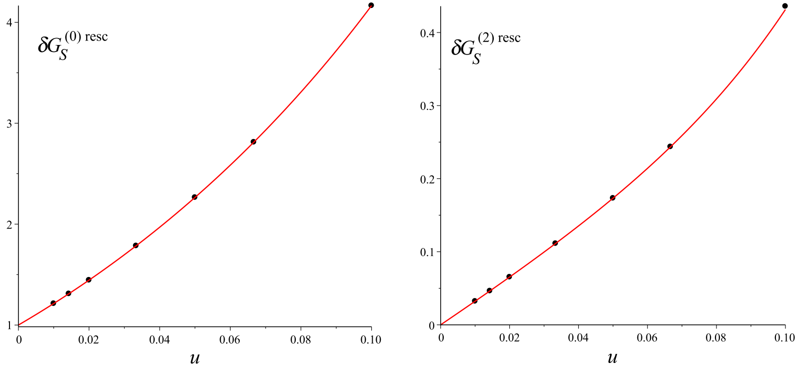


EOB dynamics for spinning bodies

[Bini, Damour & Geralico 2016]



$$\left.\begin{array}{l}\text{First law for spinning bodies}\\ \text{GSF contribution to redshift}\end{array}\right\} \Longrightarrow \text{SO coupling function } \delta G_S(u,\hat{a})$$