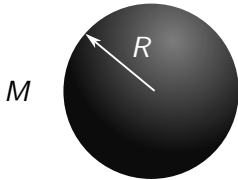# Tidal Love numbers of spinning black holes

Alexandre Le Tiec

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Observatoire de Paris / CNRS
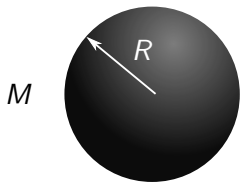
Collaborators: M. Casals & E. Franzin
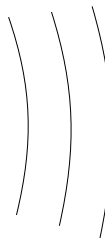
Newtonian theory of Love numbers

$$U = \frac{M}{r}$$

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$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(\mathbf{0})$$

$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab}$$

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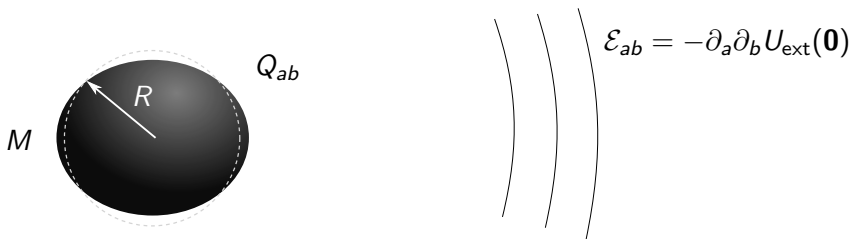
$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(\mathbf{0})$$

$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} + \frac{3}{2} \frac{x^a x^b Q_{ab}}{r^5}$$

Newtonian theory of Love numbers

$$Q_{ab} = \lambda_2 \mathcal{E}_{ab}$$

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Newtonian theory of Love numbers

$$Q_{ab} = \lambda_2 \mathcal{E}_{ab}$$
$$= -\frac{2}{3} k_2 R^5 \mathcal{E}_{ab}$$

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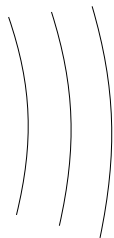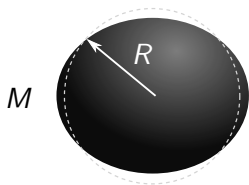$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(\mathbf{0})$$

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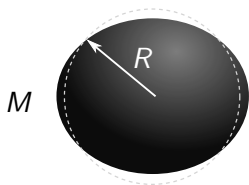

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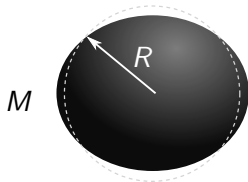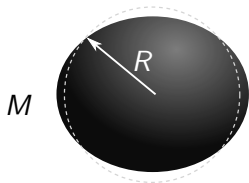

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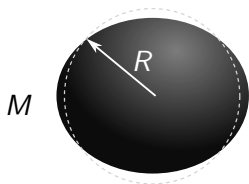
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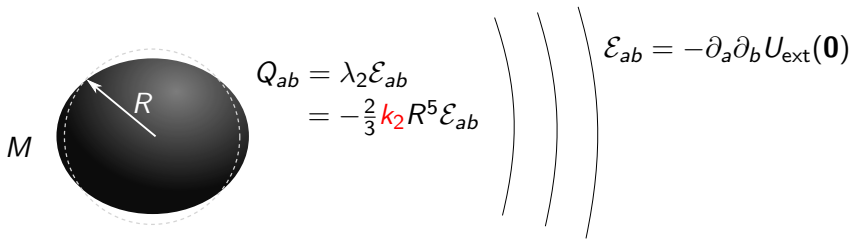
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Tidal Love numbers $k_\ell \longleftrightarrow$ **body's internal structure**

Relativistic theory of Love numbers

- Electric-type and magnetic-type tidal moments:

$$\mathcal{E}_{a_1 \cdots a_\ell} \propto (C_{0a_1 0a_2; a_3 \cdots a_\ell})_{\mathsf{STF}} \quad \text{and} \quad \mathcal{B}_{a_1 \cdots a_\ell} \propto (\varepsilon_{a_1 bc} C_{a_2 0bc; a_3 \cdots a_\ell})_{\mathsf{STF}}$$

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- Metric and Geroch-Hansen multipole moments:

  $$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + \underbrace{h_{\alpha\beta}^{\text{tidal}}}_{\sim r^\ell} + \underbrace{h_{\alpha\beta}^{\text{resp}}}_{\sim r^{-(\ell+1)}} \quad \longrightarrow \quad \left\{ \begin{array}{l} M_{\ell m} = \mathring{M}_{\ell m} + \delta M_{\ell m} \\ S_{\ell m} = \mathring{S}_{\ell m} + \delta S_{\ell m} \end{array} \right.$$

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- Four families of tidal deformability parameters:

  $$\lambda^{M\mathcal{E}}_{\ell m} \equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{E}_{\ell m}} \qquad \lambda^{S\mathcal{B}}_{\ell m} \equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{B}_{\ell m}}$$

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- Dimensionless tidal Love numbers:

  $$k^{M\mathcal{E}}_{\ell m} \equiv -\frac{(2\ell-1)!!}{2(\ell-2)!} \frac{\lambda^{M\mathcal{E}}_{\ell m}}{R^{2\ell+1}}$$

Black holes have *zero* Love numbers

Reference	Background	Tidal field
[Binnington & Poisson 2009]	Schwarzschild	weak, generic ℓ
[Damour & Nagar 2009]	Schwarzschild	weak, generic ℓ
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $\ell = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(a)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(a^2)$	weak, $(\ell, m) = (2, 0)$

Problem of **fine-tuning** from an Effective-Field-Theory perspective

Investigating Kerr's Love
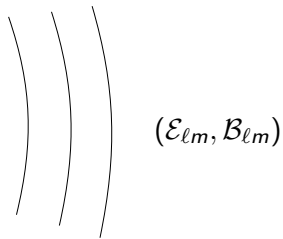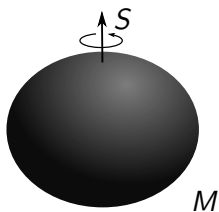
$(\mathcal{E}_{\ell m}, \mathcal{B}_{\ell m}) \to \psi_0 \to \Psi \to h_{\alpha\beta} \to (M_{\ell m}, S_{\ell m}) \to \lambda_{\ell m}^{M/S, \mathcal{E}/\mathcal{B}}$

Metric reconstruction through the Hertz potential Ψ

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c \to \infty} \psi_0^{\ell m} \propto \mathcal{E}_{\ell m} \, r^{\ell-2} \left[1 + 2k_{\ell m} \left(R/r \right)^{2\ell+1} \right] {}_2Y_{\ell m}(\theta, \phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{\ell m} \propto \left[\mathcal{E}_{\ell m} + \frac{3i}{\ell+1} \, \mathcal{B}_{\ell m} \right] R_{\ell m}(r) \, {}_2Y_{\ell m}(\theta, \phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{\ell m}(r) = \underbrace{r^{\ell-2} \left(1 + \cdots \right)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + \kappa_{\ell m} \underbrace{r^{-\ell-3} \left(1 + \cdots \right)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

To Love or not to Love?

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Eric is not in Love

- The growing solution $R_{\ell m}^{\text{tidal}}$ is not unique
- Specify it *uniquely* by requiring its **smoothness**
- Since $R_{\ell m}$ is smooth, we conclude that $\kappa_{\ell m} = 0$

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Marc and I are in Love

- **Analytic continuation** of $\ell \in \mathbb{R}$
- The growing/decaying solutions are specified *uniquely*
- Smoothness of $R_{\ell m}$ yields a response coefficient $\kappa_{\ell m} \neq 0$

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Edgardo has mixed feelings

Kerr black hole linear response

$$R_{\ell m}(r) = \underbrace{R_{\ell m}^{\text{tidal}}(r)}_{\sim r^{\ell-2}} + 2k_{\ell m}\underbrace{R_{\ell m}^{\text{resp}}(r)}_{\sim r^{-(\ell+3)}}$$

- The coefficients $k_{\ell m}$ are related to the Love numbers and read

$$k_{\ell m} = -\frac{i}{4\pi}\sinh(2\pi m\gamma)\,|\Gamma(\ell+1+2im\gamma)|^2\,\frac{(\ell-2)!(\ell+2)!}{(2\ell)!(2\ell+1)!}$$

- The linear response vanishes identically when:
  - the black hole spin vanishes ($\gamma = 0$)
  - the tidal field is axisymmetric ($m = 0$)

- Reconstruct the Kerr black hole response $h_{\alpha\beta}^{\text{resp}}$ via Ψ^{resp}

Love numbers of a Kerr black hole

- We compute the Love numbers to linear order in $\chi \equiv a/M$

- The modes of the mass/current quadrupole moments are

$$\delta M_{2m} = \frac{im\chi}{180} (2M)^5 \, \mathcal{E}_{2m} \quad \text{and} \quad \delta S_{2m} = \frac{im\chi}{180} (2M)^5 \, \mathcal{B}_{2m}$$

- The associated dimensionless tidal Love numbers are

$$k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} = -\frac{im\chi}{120} \quad \text{and} \quad k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} = 0$$

- For a dimensionless black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \simeq 2 \times 10^{-3} \quad \longrightarrow \quad \text{black holes are \textbf{"stiff"}}$$

Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\text{el}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\text{mag}} \mathcal{B}_{ab}$$

- For a spinning black hole we have the more general tensorial relations

$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}_{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}_{cd}$$

- The effective description of spinning black holes requires augmenting the p.p. action by the nonminimal couplings

$$S \supset -\frac{1}{4} \int ds \, \lambda_{abcd} \left(\mathcal{E}_{ab} \mathcal{E}_{cd} + \mathcal{B}_{ab} \mathcal{B}_{cd} \right)$$
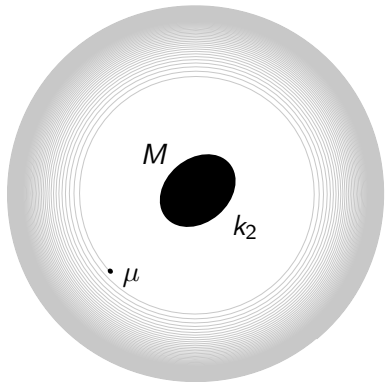
Love tensor of a Kerr black hole

$$\lambda_{abcd} = \frac{\chi}{180} (2M)^5 \begin{pmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} & \mathbf{I}_{13} \\ \mathbf{I}_{12} & -\mathbf{I}_{11} & \mathbf{I}_{23} \\ \mathbf{I}_{13} & \mathbf{I}_{23} & \mathbf{0} \end{pmatrix}$$

$$\mathbf{I}_{11} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{I}_{12} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{I}_{13} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \mathbf{I}_{23} \equiv \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

Observing black hole tidal deformability

[Pani & Maselli, IJMPD 2019]

Accumulated GW phase in LISA
band during quasi-circular inspiral
down to Schwarzschild ISCO:

$$\Phi_{\text{tidal}} \simeq -80 \left(\frac{10^{-7}}{\mu/M}\right)\left(\frac{k_2}{0.002}\right)$$

$\uparrow$

like 1st order dissipative GSF

Summary

- Love numbers of Kerr black holes do not vanish in general

- We computed in closed-form the leading (quadrupolar) Love numbers to linear order in the black hole spin

- Kerr black holes deform like any other self-gravitating body, despite being particularly "stiff" compact objects

- The black hole tidal deformation contribution to the GW phase of EMRIs will be detectable by LISA

- New black hole test of the Kerr-like nature of the massive compact objects at the center of galaxies

Spinning black holes fall in Love!

Why analytic continuation?

$$R_{\ell m}(r) = \underbrace{r^{\ell-2}\,(1+\cdots)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + \kappa_{\ell m}\,\underbrace{r^{-\ell-3}\,(1+\cdots)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Ambiguity in the linear response [Fang & Lovelace 2005; Gralla 2018]

The decaying solution $R_{\ell m}^{\text{resp}}$ is affected by a radial coord. transfo.

Ambiguity in the tidal field [Pani, Gualtieri, Maselli & Ferrari 2015]

The growing solution $R_{\ell m}^{\text{tidal}} + \alpha R_{\ell m}^{\text{resp}}$ still qualifies as a tidal solution

Two basis of independent solutions

- Dimensionless radial coordinate and spin parameter

$$x \equiv \frac{r - r_+}{r_+ - r_-} \quad \text{and} \quad \gamma = \frac{a}{r_+ - r_-}$$

- Smooth and unsmooth solutions:

$$R_{\ell m}^{\text{smooth}} = x^{-2}(1+x)^{-2} \, \mathbf{F}(-\ell-2, \ell-1, -1+2im\gamma; -x)$$
$$R_{\ell m}^{\text{unsmooth}} = (1+1/x)^{2im\gamma} \, \mathbf{F}(-\ell+2, \ell+3, 3-2im\gamma; -x)$$
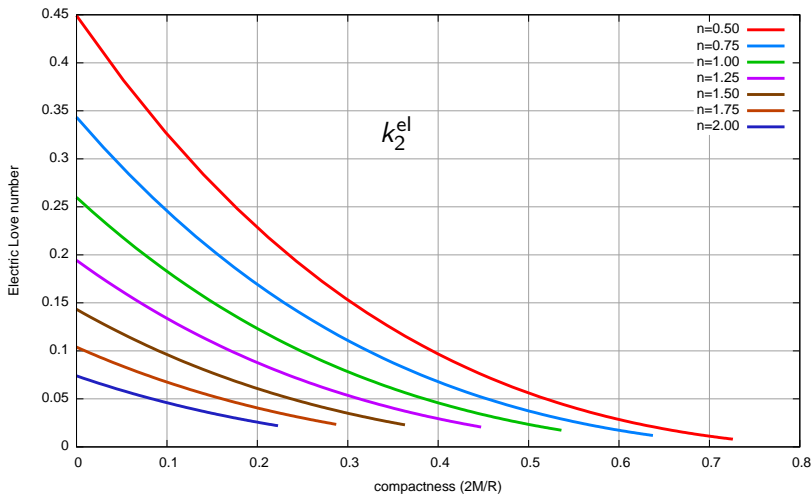
- Tidal field and linear response solutions:

$$R_{\ell m}^{\text{tidal}} = \frac{x^\ell}{(1+x)^2} \, F(-\ell-2, -\ell-2im\gamma, -2\ell; -1/x) \sim x^{\ell-2}$$
$$R_{\ell m}^{\text{resp}} = \frac{x^{-\ell-1}}{(1+x)^2} \, F(\ell-1, \ell+1-2im\gamma, 2\ell+2; -1/x) \sim x^{-\ell-3}$$
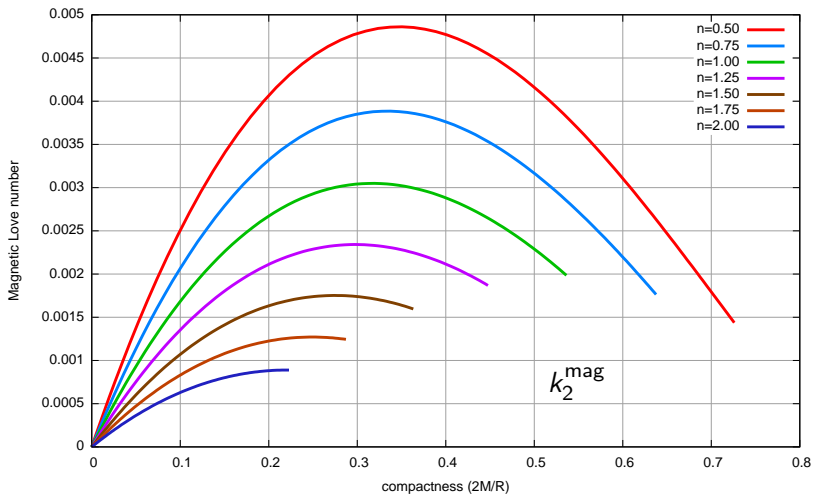
Love numbers of neutron stars

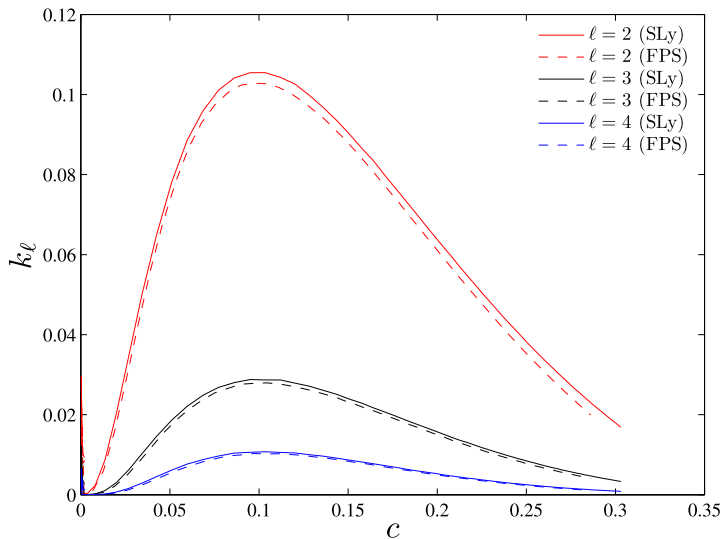# Love numbers of neutron stars
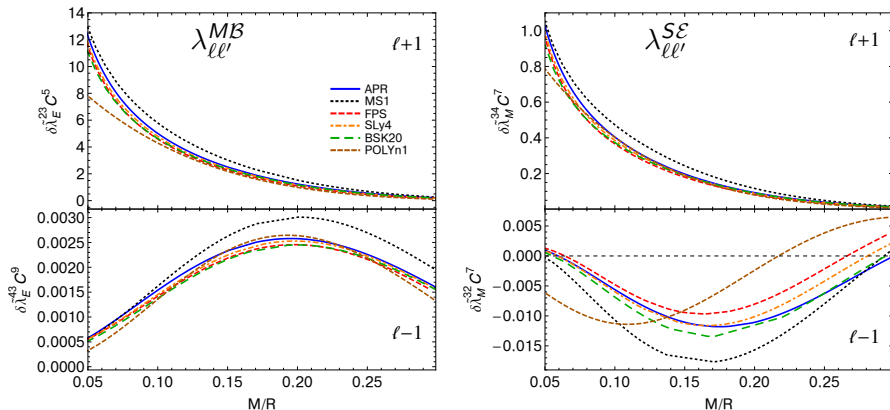
[Binnington & Poisson, PRD 2009]

Love numbers of neutron stars

[Damour & Nagar, PRD 2009]

Love numbers of *spinning* neutron stars

[Pani, Gualtieri & Ferrari, PRD 2015]

Restricted to an *axisymmetric* tidal perturbation ($m = m' = 0$)

Gravitational-wave observations

[Chatziioannou, GRG 2020]

$\Lambda \equiv \lambda_2/M^5 \propto k_2/C^5$