Tidal Love numbers of Kerr black holes

Alexandre Le Tiec

Laboratoire Univers et Théories
Observatoire de Paris / CNRS

Collaborators: M. Casals & E. Franzin

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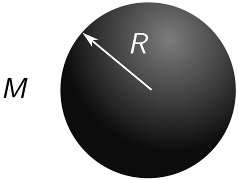
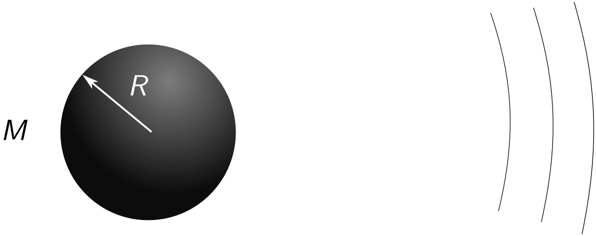
Newtonian theory of Love numbers



$$U = \frac{M}{r}$$

Newtonian theory of Love numbers



$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(\mathbf{0})$$

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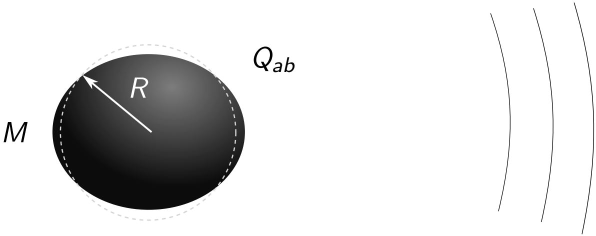

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$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} + \frac{3}{2} \frac{x^a x^b Q_{ab}}{r^5}$$

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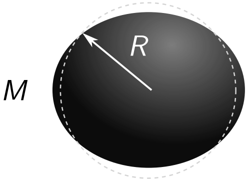

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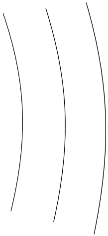

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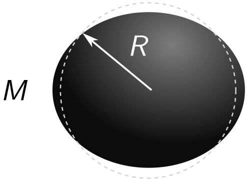

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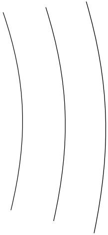

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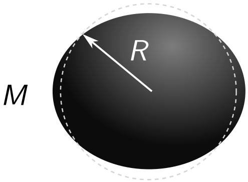

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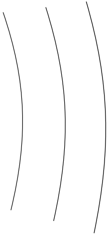

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Newtonian theory of Love numbers

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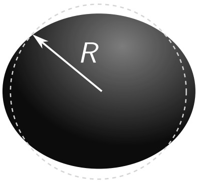
M



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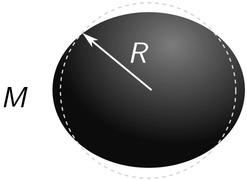

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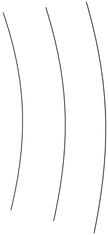

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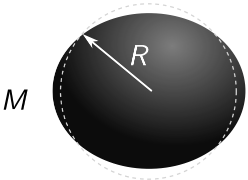
Newtonian theory of Love numbers



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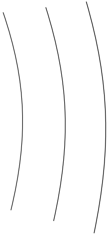

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$$\psi_0 = \sum_{\ell \geqslant 2} \sum_{|m| \leqslant \ell} \sqrt{\frac{(\ell+2)(\ell+1)}{\ell(\ell-1)}}\, r^{\ell-2}\, \mathcal{E}_{\ell m} \left[1 + 2 k_\ell \left(\frac{R}{r} \right)^{2\ell+1} \right] {}_2Y_{\ell m}$$

Newtonian theory of Love numbers

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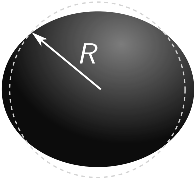

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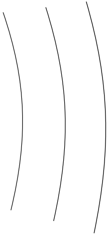

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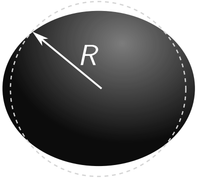

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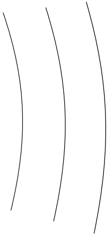

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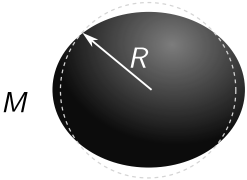
Tidal Love numbers $k_{\ell m} \longleftrightarrow$ **body's internal structure**

Newtonian theory of Love numbers



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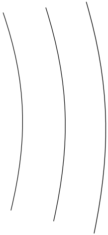

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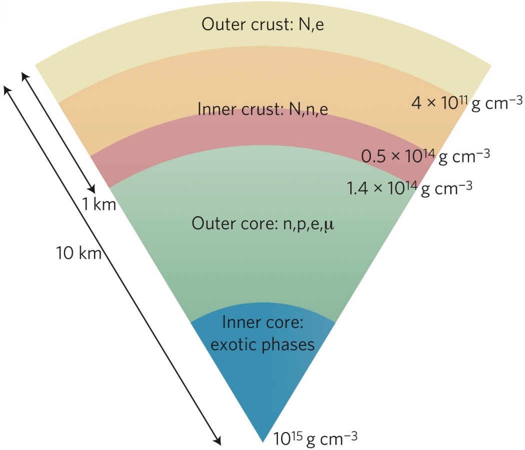
$$k_{\ell m} = k_\ell^{(0)} + im\chi\, k_\ell^{(1)} + m^2\chi^2\, k_\ell^{(2)} + im^3\chi^3\, k_\ell^{(3)} + \cdots$$

Internal structure of neutron stars

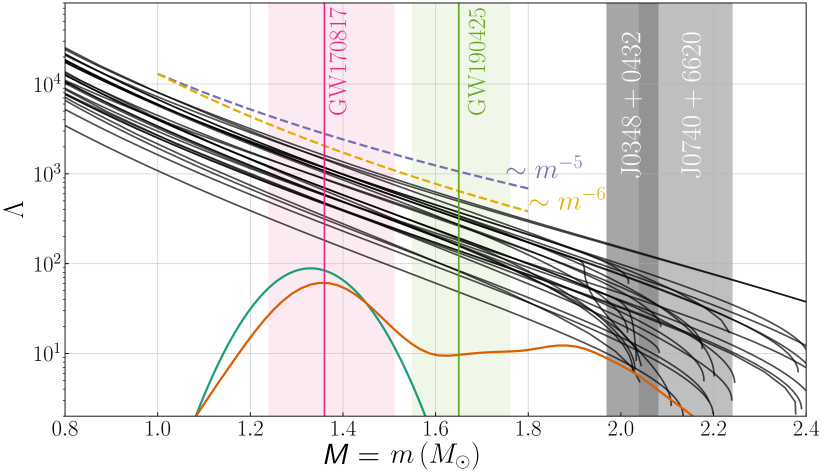


GW observations as probes of **neutron star internal structure**

Gravitational-wave observations

[Chatziioannou, GRG 2020]

$\Lambda \equiv \lambda_2 / M^5 \propto k_2 / C^5$



Relativistic theory of Love numbers

- Electric-type and magnetic-type tidal moments:

$$\mathcal{E}_L \propto (C_{0a_1 0a_2;a_3\cdots a_\ell})_{\mathsf{STF}} \quad \text{and} \quad \mathcal{B}_L \propto (\varepsilon_{a_1 bc} C_{a_2 0bc;a_3\cdots a_\ell})_{\mathsf{STF}}$$

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- Two families of tidal deformability parameters:

$$\delta M_L = \lambda_\ell^{\text{el}} \mathcal{E}_L \quad \text{and} \quad \delta S_L = \lambda_\ell^{\text{mag}} \mathcal{B}_L$$

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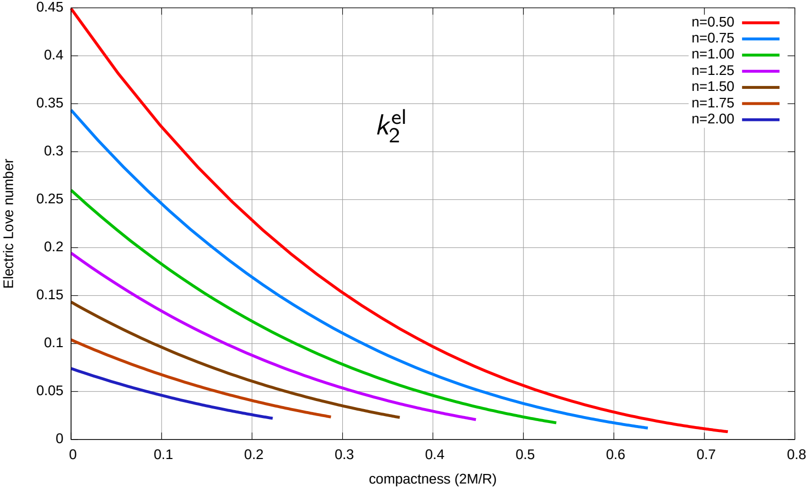
- Dimensionless tidal Love numbers:

$$k^{\mathrm{el/mag}}_\ell \equiv -\frac{(2\ell-1)!!}{2(\ell-2)!}\,\frac{\lambda^{\mathrm{el/mag}}_\ell}{R^{2\ell+1}}$$

Love numbers of neutron stars

[Binnington & Poisson, PRD 2009]



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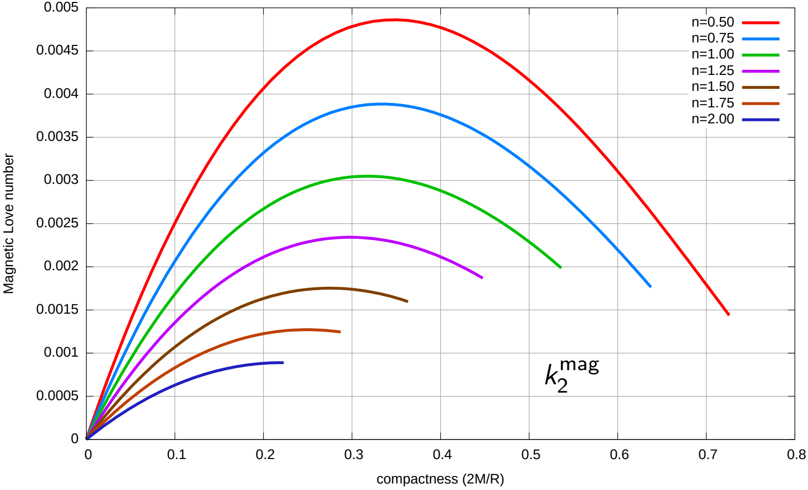

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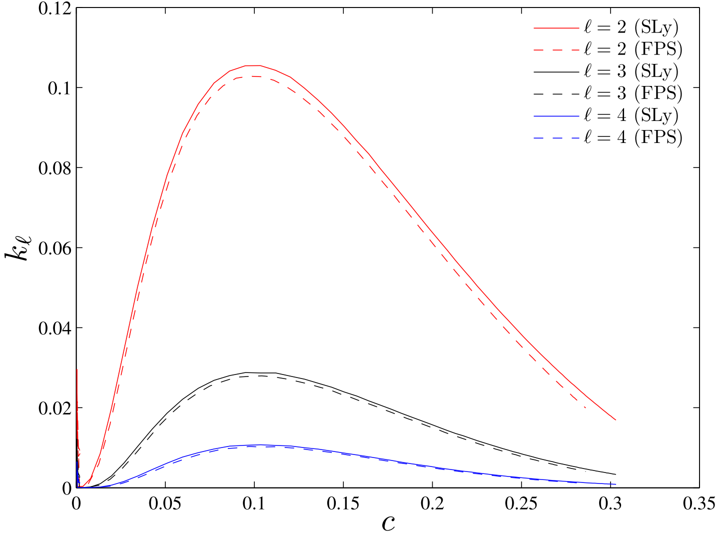
[Damour & Nagar, PRD 2009]



Love numbers of *spinning* compact objects

- The spin breaks the spherical symmetry of the background
  - No proportionality between $(\delta M_L, \delta S_L)$ and $(\mathcal{E}_L, \mathcal{B}_L)$
  - Degeneracy of the azimuthal number m lifted
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- Four families of tidal deformability parameters:

$$\lambda^{M\mathcal{E}}_{\ell\ell' mm'} \equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{E}_{\ell' m'}} \qquad \lambda^{S\mathcal{B}}_{\ell\ell' mm'} \equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{B}_{\ell' m'}}$$

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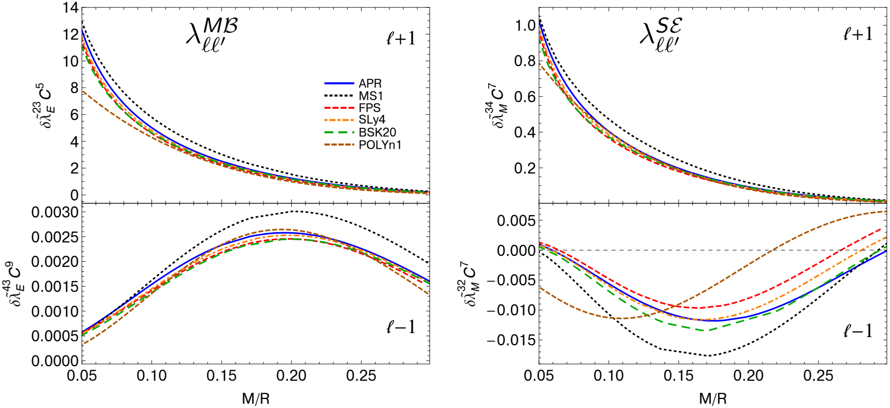
Love numbers of *spinning* neutron stars

[Pani, Gualtieri & Ferrari, PRD 2015]



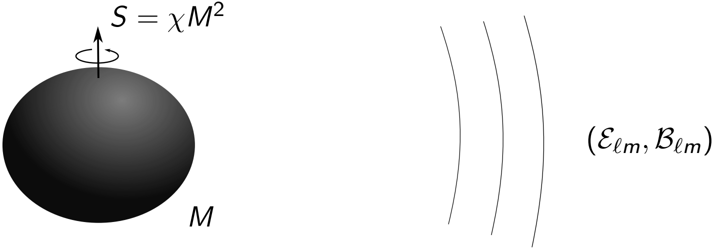
Restricted to an *axisymmetric* tidal perturbation ($m = m' = 0$)

Black holes have *zero* Love numbers

Reference	**Background**	**Tidal field**
[Binnington & Poisson 2009]	Schwarzschild	weak, generic ℓ
[Damour & Nagar 2009]	Schwarzschild	weak, generic ℓ
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $\ell = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(S^2)$	weak, $(\ell, m) = (2, 0)$

Problem of **fine-tuning** from an Effective-Field-Theory perspective

Investigating Kerr's Love



$$(\mathcal{E}_{\ell m}, \mathcal{B}_{\ell m}) \to \psi_0 \to \Psi \to h_{\alpha\beta} \to (M_{\ell m}, S_{\ell m}) \to \lambda_{\ell m}^{M/S, \mathcal{E}/\mathcal{B}}$$

Metric reconstruction through the Hertz potential Ψ

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c\to\infty} \psi_0^{\ell m} \propto \mathcal{E}_{\ell m}\, r^{\ell-2} \left[1 + 2k_{\ell m}\,(R/r)^{2\ell+1}\right] {}_2Y_{\ell m}(\theta,\phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{\ell m} \propto \left[\mathcal{E}_{\ell m} + \tfrac{3i}{\ell+1}\,\mathcal{B}_{\ell m}\right] R_{\ell m}(r)\; {}_2Y_{\ell m}(\theta,\phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{\ell m}(r) = \underbrace{r^{\ell-2}\,(1+\cdots)}_{\text{tidal field }\; R^{\text{tidal}}_{\ell m}} + \kappa_{\ell m}\, \underbrace{r^{-\ell-3}\,(1+\cdots)}_{\text{linear response }\; R^{\text{resp}}_{\ell m}}$$

To Love or not to Love?

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Eric Poisson is not in Love

- The growing solution $R^{\text{tidal}}_{\ell m}$ is not unique
- Specify it *uniquely* by requiring its **smoothness**
- Since $R_{\ell m}$ is smooth, he concludes that $\kappa_{\ell m} = 0$

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Marc Casals and I are in Love

- **Analytic continuation** of $\ell \in \mathbb{R}$
- The growing/decaying solutions are specified *uniquely*
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Edgardo Franzin has mixed feelings

Why analytic continuation?

$$R_{\ell m}(r) = \underbrace{r^{\ell-2}\,(1+\cdots)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + \kappa_{\ell m}\,\underbrace{r^{-\ell-3}\,(1+\cdots)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Ambiguity in the linear response [Fang & Lovelace 2005; Gralla 2018]

The decaying solution $R_{\ell m}^{\text{resp}}$ is affected by a radial coord. transfo.

Ambiguity in the tidal field [Pani, Gualtieri, Maselli & Ferrari 2015]

The growing solution $R_{\ell m}^{\text{tidal}} + \alpha R_{\ell m}^{\text{resp}}$ still qualifies as a tidal solution

Kerr black hole linear response

$$R_{\ell m}(r) \;=\; \underbrace{R^{\text{tidal}}_{\ell m}(r)}_{\sim r^{\ell-2}} + 2{\color{red}k_{\ell m}}\,\underbrace{R^{\text{resp}}_{\ell m}(r)}_{\sim r^{-(\ell+3)}}$$

- The coefficients ${\color{red}k_{\ell m}}$ can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$${\color{red}k_{\ell m}} = -im\gamma\,\frac{(\ell-2)!(\ell+2)!}{2(2\ell)!(2\ell+1)!}\,\prod_{n=0}^{\ell}(n^2+4m^2\gamma^2)\,, \quad \gamma = a/(r_+-r_-)$$

- The linear response vanishes identically when:
  - the black hole spin vanishes (${\color{red}\gamma} = 0$)
  - the tidal field is axisymmetric (${\color{red}m} = 0$)
- Reconstruct the Kerr black hole response $h^{\text{resp}}_{\alpha\beta}$ via Ψ^{resp}

Love numbers of a Kerr black hole

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- For a dimensionless black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \simeq 2 \times 10^{-3} \quad \longrightarrow \quad \text{black holes are } \textbf{“rigid”}$$

Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\mathrm{el}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\mathrm{mag}} \mathcal{B}_{ab}$$

- For a spinning black hole we have the more general tensorial relations

$$\delta M_{ab} \doteq \lambda_{abcd} \mathcal{E}_{cd} \quad \text{and} \quad \delta S_{ab} \doteq \lambda_{abcd} \mathcal{B}_{cd}$$

- The tidal Love tensor of a Kerr black hole reads

$$(\lambda_{abcd}) \doteq \frac{\chi}{180} (2M)^5 \begin{pmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} & \mathbf{I}_{13} \\ \mathbf{I}_{12} & -\mathbf{I}_{11} & \mathbf{I}_{23} \\ \mathbf{I}_{13} & \mathbf{I}_{23} & \mathbf{0} \end{pmatrix}$$

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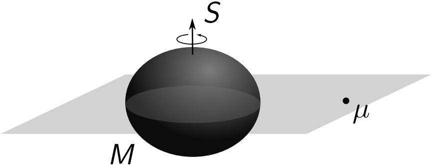
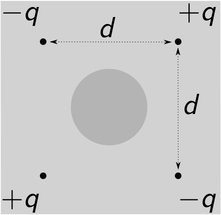
$$\mathbf{I}_{11} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{I}_{12} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{I}_{13} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \mathbf{I}_{23} \equiv \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

Newtonian static quadrupolar tide





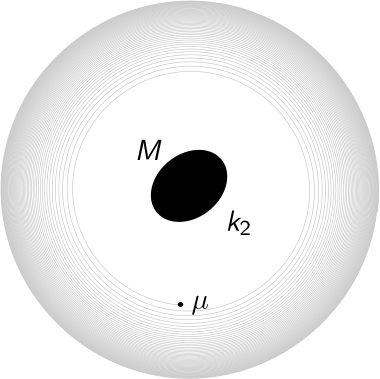
$$\mathcal{E}_{ab} = \frac{\mu}{r^3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\delta M_{ab} \doteq \underset{\uparrow}{3Q}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tfrac{\chi}{180}\,(2M)^5\,\tfrac{\mu}{r^3} = qd^2$$

Observing black hole tidal deformability

[Pani & Maselli, IJMPD 2019]



Accumulated GW phase in LISA band during quasi-circular inspiral down to Schwarzschild ISCO:

$$\Phi_{\rm tidal} \simeq -2 \times 10^3 \left(\frac{10^{-7}}{\mu/M}\right)\left(\frac{k_2}{0.002}\right)$$

$\uparrow$

like 1st order dissipative self-force

Summary

- Love numbers of Kerr black holes do not vanish in general
- We computed in closed-form the leading (quadrupolar) Love numbers to linear order in the black hole spin
- Kerr black holes deform like any other self-gravitating body, despite being particularly “rigid” compact objects
- The black hole tidal deformation contribution to the GW phase of EMRIs could be detectable by LISA
- New black hole test of the Kerr-like nature of the massive compact objects at the center of galaxies

Spinning black holes fall in Love!

Two basis of independent solutions

- Dimensionless radial coordinate and spin parameter:

$$x \equiv \frac{r - r_+}{r_+ - r_-} \quad \text{and} \quad \gamma = \frac{a}{r_+ - r_-}$$

- Smooth and unsmooth solutions:

$$R_{\ell m}^{\text{smooth}} = x^{-2}(1+x)^{-2}\, \mathbf{F}(-\ell - 2, \ell - 1, -1 + 2im\gamma; -x)$$
$$R_{\ell m}^{\text{unsmooth}} = (1 + 1/x)^{2im\gamma}\, \mathbf{F}(-\ell + 2, \ell + 3, 3 - 2im\gamma; -x)$$

- Tidal and response solutions:

$$R_{\ell m}^{\text{tidal}} = \frac{x^{\ell}}{(1+x)^2}\, F(-\ell - 2, -\ell - 2im\gamma, -2\ell; -1/x) \sim x^{\ell - 2}$$
$$R_{\ell m}^{\text{resp}} = \frac{x^{-\ell-1}}{(1+x)^2}\, F(\ell - 1, \ell + 1 - 2im\gamma, 2\ell + 2; -1/x) \sim x^{-\ell-3}$$