First laws of compact binary mechanics

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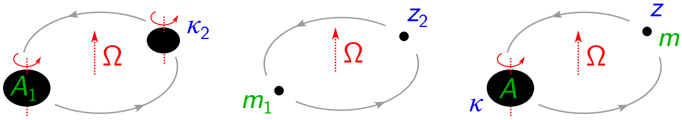
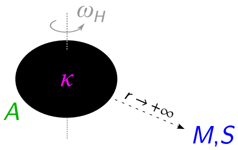


The laws of black hole mechanics

[Hawking 1972, Bardeen, Carter & Hawking 1973]

- Zeroth law of mechanics:

$$\kappa = \text{const.} \quad (\text{on } \mathcal{H})$$

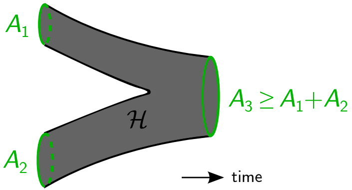


- First law of mechanics:

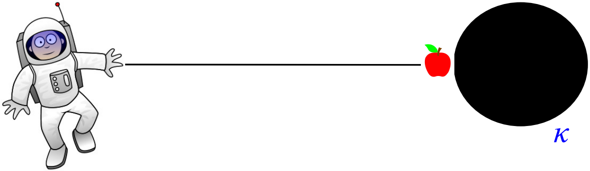
$$\delta M = \omega_H \, \delta S + \frac{\kappa}{8\pi} \, \delta A$$

- Second law of mechanics:

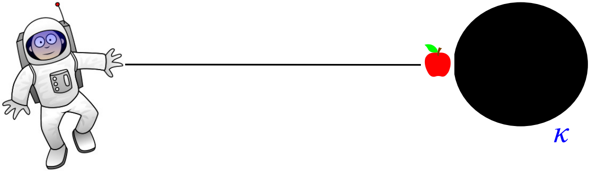
$$\delta A \geq 0$$



What is the horizon surface gravity?



What is the horizon surface gravity?



- For an event horizon $\mathcal{H}$ generated by a Killing field k^α:

$$\kappa^2 \equiv \frac{1}{2} \left(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha\right)\Big|_{\mathcal{H}}$$

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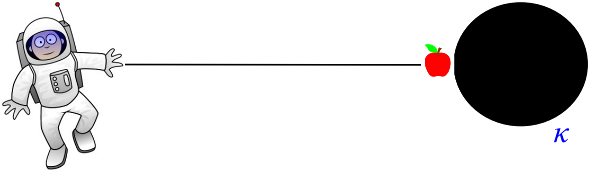

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- For a Schwarzschild black hole of mass M, this yields

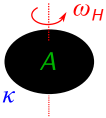
$$\kappa = \frac{1}{4M} = \frac{GM}{R_S^2}$$

Outline

1. Circular-orbit binaries: geometrical methods

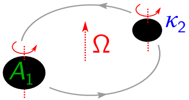
2. Beyond circular motion: Hamiltonian methods

First laws of compact binary mechanics



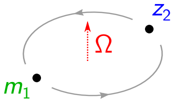
$$\delta M - \omega_H \, \delta S = \frac{\kappa}{8\pi} \, \delta A$$

[Bardeen *et al.* 1973]



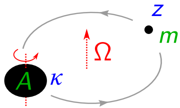
$$\delta M - \Omega \, \delta J = \sum_a \frac{\kappa_a}{8\pi} \, \delta A_a$$

[Friedman *et al.* 2002]



$$\delta M - \Omega \, \delta J = \sum_a z_a \, \delta m_a$$

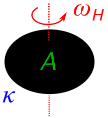
[Le Tiec *et al.* 2012]
[Blanchet *et al.* 2013]



$$\delta M - \Omega \, \delta J = \frac{\kappa}{8\pi} \, \delta A + z \, \delta m$$

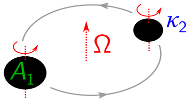
[Gralla & Le Tiec 2013]

First laws of compact binary mechanics



$$\delta M - \omega_H \, \delta S = 4\mu\kappa \, \delta\mu$$

[Bardeen *et al.* 1973]



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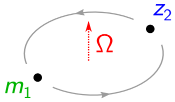

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[Le Tiec *et al.* 2012]

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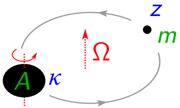
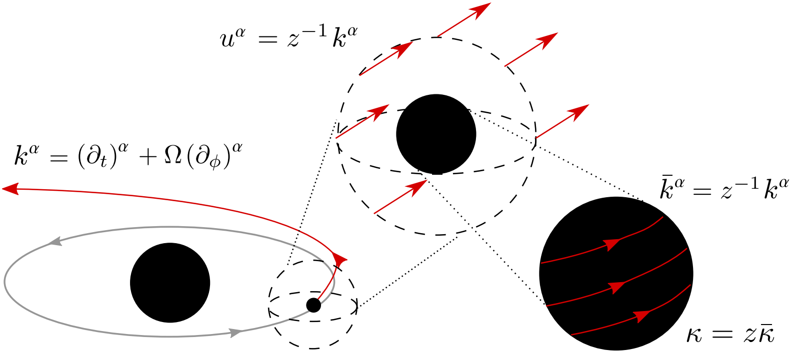


$$\delta M - \Omega \, \delta J = 4\mu\kappa \, \delta\mu + z \, \delta m$$

[Gralla & Le Tiec 2013]

Surface gravity and redshift variable

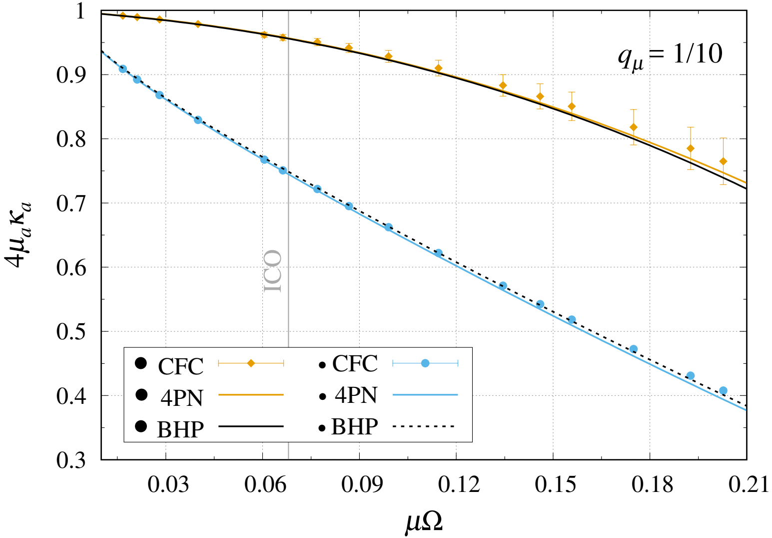
[Pound (unpublished)]



(Credit: Zimmerman, Lewis & Pfeiffer 2016)

Surface gravity *vs* orbital frequency

[Le Tiec & Grandclément 2018]



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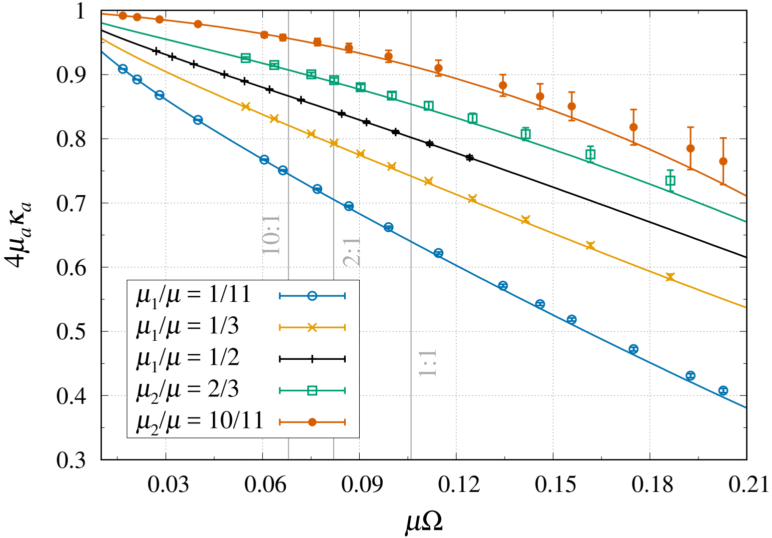
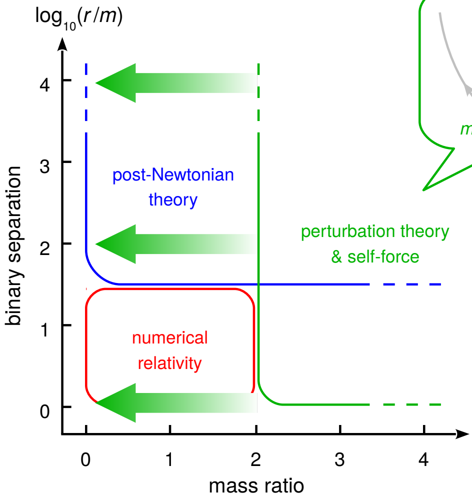
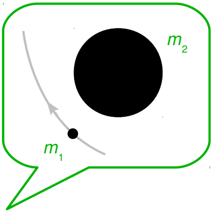


Perturbation theory for comparable masses





$$q = m_1 / m_2$$
$$\downarrow$$
$$\nu = m_1 m_2 / m^2$$

Perturbation theory for comparable masses

Comparisons to numerical relativity

dissipative

- Recoil velocity [Fitchett & Detweiler 1984, Nagar 2013]
- Head-on waveform [Anninos *et al.* 1995, Sperhake *et al.* 2011]
- Inspiral waveform [van de Meent & Pfeiffer 2020, Rifat *et al.* 2020]

conservative

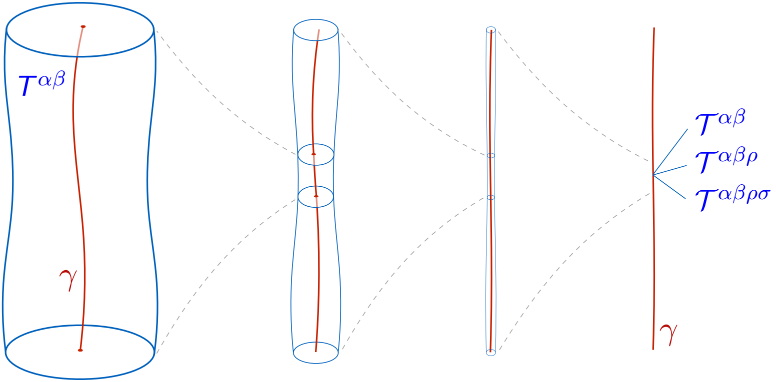
- Periastron advance [Le Tiec *et al.* 2011, 2013]
- Binding energy [Le Tiec, Buonanno & Barausse 2012]
- Surface gravity [Zimmerman *et al.* 2016, Le Tiec & Grandclément 2018]

Structure of Einstein equation

- Polynomial nonlinearity using geometric variables [Harte 2014]
- Exact EOB energy map to $O(G)$ [Damour 2016, Vines 2017]
- **Link to double copy?**

Multipolar gravitational skeleton

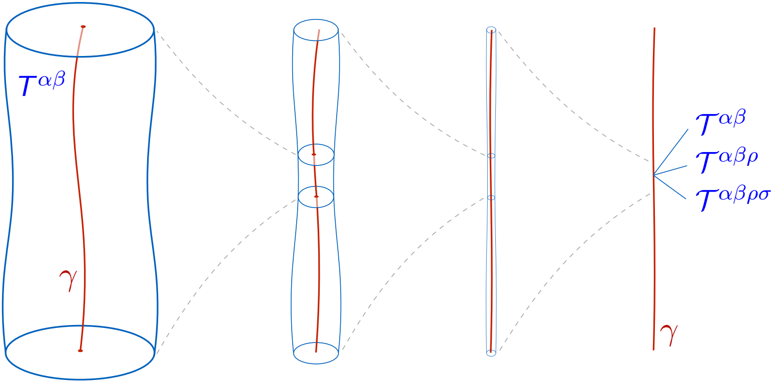
[Mathisson 1937, Tulczyjew 1957]



$$T^{\alpha\beta} \to T^{\alpha\beta}_{\mathsf{skel}} = \int_{\gamma} \mathrm{d}\tau \left[\underbrace{\mathcal{T}^{\alpha\beta}\,\delta_4}_{\text{monopole}} + \underbrace{\nabla_\rho \left(\mathcal{T}^{\alpha\beta\rho}\,\delta_4\right)}_{\text{dipole}} + \cdots \right]$$

Multipolar gravitational skeleton

[Mathisson 1937, Tulczyjew 1957]



$$T^{\alpha\beta} \to T^{\alpha\beta}_{\mathsf{skel}} = \int_\gamma \mathrm{d}\tau \left[\underbrace{u^{(\alpha} p^{\beta)} \, \delta_4}_{\text{monopole}} + \underbrace{\nabla_\rho \left(u^{(\alpha} S^{\beta)\rho} \, \delta_4 \right)}_{\text{dipole}} + \cdots \right]$$

Quadrupolar particles on a circular orbit

[Ramond, Le Tiec & Noûs 2020]

- Helical Killing field k^α so that

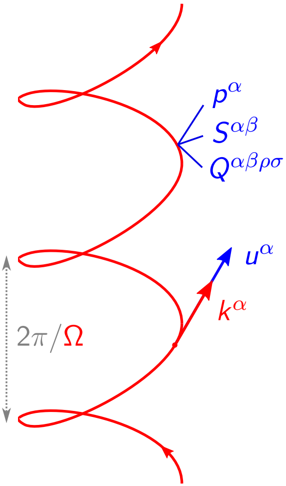
$$\mathcal{L}_k g_{\alpha\beta} = 0$$

- Each particle worldline γ is an integral curve of k^α:

$$k^\alpha|_\gamma = z\, u^\alpha$$

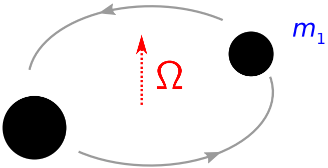
- The particle multipoles are all Lie-dragged along k^α:

$$\mathcal{L}_k p^\alpha = \mathcal{L}_k S^{\alpha\beta} = \mathcal{L}_k Q^{\alpha\beta\rho\sigma} = 0$$



First law with leading finite-size effects

[Ramond, Le Tiec & Noûs (in progress)]



$$\delta M - \Omega\, \delta J = \sum_a |k|_a\, \delta m_a$$

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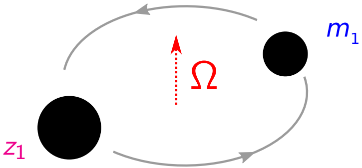

$$\delta M - \Omega\, \delta J = \sum_a z_a\, \delta m_a$$

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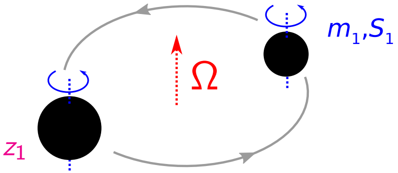
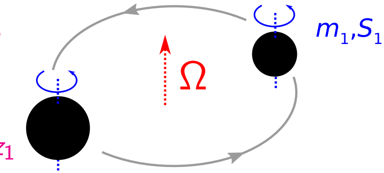


$$\delta M - \Omega\,\delta J = \sum_a z_a\,\delta m_a - \sum_a |\nabla k|_a\,\delta S_a$$

First law with leading finite-size effects

[Ramond, Le Tiec & Noûs (in progress)]

$\dot{\mathbf{S}}_a = \boldsymbol{\Omega}_a \times \mathbf{S}_a$



$$\delta M - \Omega\,\delta J = \sum_a z_a\,\delta m_a - \sum_a (\Omega - \Omega_a)\,\delta S_a$$

First law with leading finite-size effects

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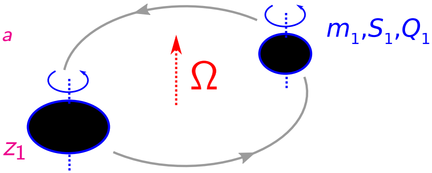

$$\delta M - \Omega\,\delta J = \sum_a z_a\,\delta m_a - \sum_a (\Omega - \Omega_a)\,\delta S_a + \sum_a |\nabla\nabla k|_a\,\delta Q_a$$

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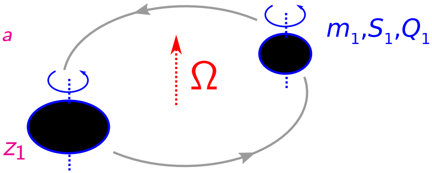

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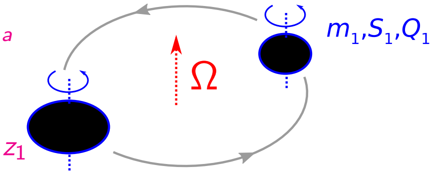
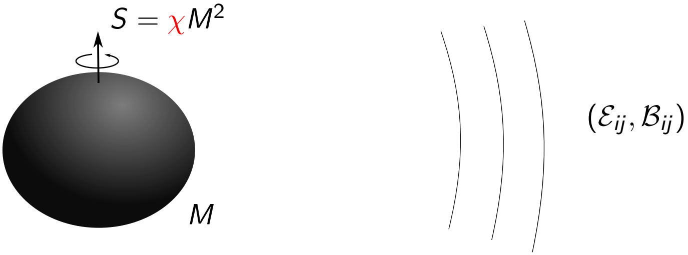


$$\delta M - \Omega\,\delta J = \sum_a z_a\,\delta m_a - \sum_a (\Omega - \Omega_a)\,\delta S_a + \sum_a \mathcal{E}_a\,\delta Q_a$$

- Spin-induced quadrupole $Q_{\text{spin}} \sim \kappa S^2$
- Tidally-induced quadrupole $Q_{\text{tidal}} \sim \lambda \mathcal{E}$

Tidal deformability of Kerr black holes

[Le Tiec & Casals 2020, Goldberger's talk]



$$Q_{ij}^{\text{tidal}} = \lambda_{ijkl}\mathcal{E}_{kl} \quad \text{where} \quad \lambda_{ijkl} \doteq \frac{\chi}{180}\,(2M)^5 \begin{pmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} & \mathbf{I}_{13} \\ \mathbf{I}_{12} & -\mathbf{I}_{11} & \mathbf{I}_{23} \\ \mathbf{I}_{13} & \mathbf{I}_{23} & \mathbf{0} \end{pmatrix}$$

Conservative tidal Love number or dissipative tidal torquing?

Outline

Averaged redshift for eccentric orbits

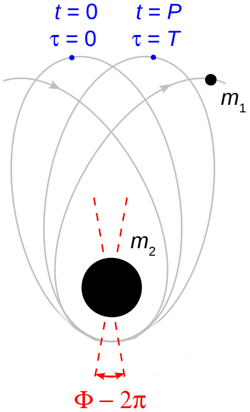
[Barack & Sago 2011]

- Generic eccentric orbit parameterized by the two requencies

$$\Omega_r = \frac{2\pi}{P}\,, \quad \Omega_\phi = \frac{\Phi}{P}$$

- Time average of redshift $z = \mathrm{d}\tau/\mathrm{d}t$ over one radial period

$$\langle z \rangle \equiv \frac{1}{P}\int_0^P z(t)\,\mathrm{d}t = \frac{T}{P}$$



First law of mechanics for eccentric orbits

[Le Tiec 2015, Blanchet & Le Tiec 2017]

- Canonical ADM Hamiltonian $H(\mathbf{x}_a, \mathbf{p}_a; m_a)$ of two point particles with constant masses m_a
- Variation δH + Hamilton's equation + orbital averaging:

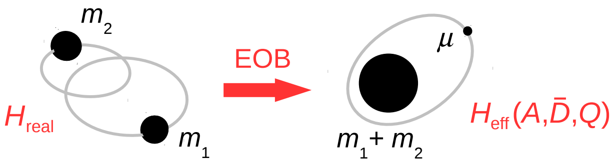
$$\delta M = \Omega_\phi \, \delta L + \Omega_r \, \delta J_r + \sum_a \langle z_a \rangle \, \delta m_a$$

- Starting at 4PN order the binary dynamics gets nonlocal in time because of gravitational-wave tails:

$$H_{\text{tail}}^{\text{4PN}}[\mathbf{x}_a(t), \mathbf{p}_a(t)] = -\frac{M}{5} \hat{I}_{ij}^{(3)}(t) \, \underset{2r}{\text{Pf}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{\tau} \, \hat{I}_{ij}^{(3)}(t+\tau)$$

- With appropriate M, L and J_r the first law still holds

EOB dynamics beyond circular motion



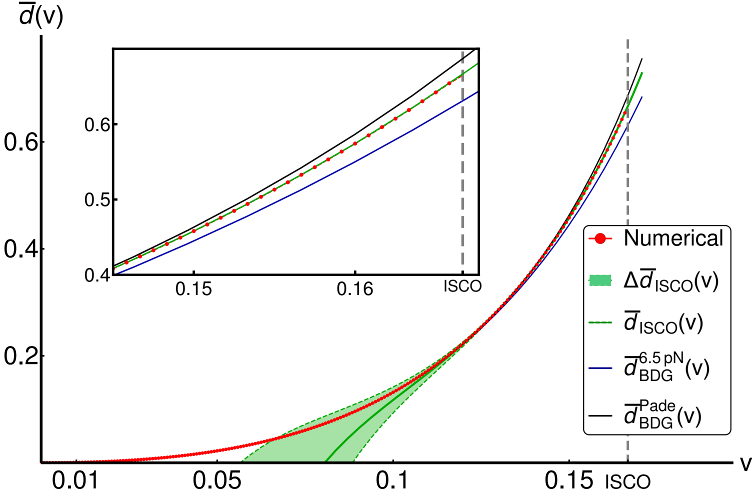
- Conservative EOB dynamics determined by "potentials"

$$
\begin{aligned}
A(r) &= 1 - 2M/r + \nu\, a(r) + \cdots \\
\bar{D}(r) &= 1 + \nu\, \bar{d}(r) + \cdots \\
Q(r) &= \nu\, q(r)\, p_r^4 + \cdots
\end{aligned}
$$

- Functions $a(r)$, $\bar{d}(r)$ and $q(r)$ controlled by $\langle z \rangle_{\text{GSF}}(\Omega_r, \Omega_\phi)$

EOB dynamics beyond circular motion

[Akcay & van de Meent 2016]



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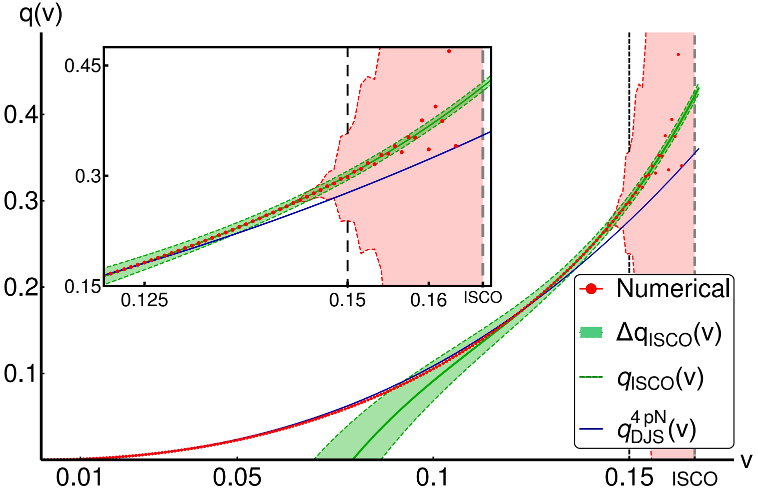

Particle Hamiltonian first law

- Geodesic motion of test mass m in Kerr geometry $\bar{g}_{\alpha\beta}$ derives from canonical Hamiltonian

$$\bar{H}(x^\mu, p_\mu) = \frac{1}{2m}\,\bar{g}^{\alpha\beta}(x)p_\alpha p_\beta$$

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$$\bar{H}(x^\mu, p_\mu) = \frac{1}{2m}\,\bar{g}^{\alpha\beta}(x) p_\alpha p_\beta$$
- Hamilton-Jacobi equation is completely separable [Carter 1968]
- Canonical transformation $(x^\mu, p_\mu) \to (q_\alpha, J_\alpha)$ to *generalized* action-angle variables [Schmidt 2002, Hinderer & Flanagan 2008]
$$\frac{\mathrm{d}J_\alpha}{\mathrm{d}\tau} = -\frac{\partial \bar{H}}{\partial q_\alpha} = 0\,, \quad \frac{\mathrm{d}q_\alpha}{\mathrm{d}\tau} = \frac{\partial \bar{H}}{\partial J_\alpha} \equiv \omega_\alpha$$

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- Varying $\bar{H}(J_\alpha)$ yields a particle Hamiltonian first law valid for *generic* bound orbits [Le Tiec 2014]

$$\delta E = \Omega_\phi\,\delta L + \Omega_r\,\delta J_r + \Omega_\theta\,\delta J_\theta + \langle z\rangle\,\delta m$$

Including conservative self-force effects

[Fujita, Isoyama, Le Tiec, Nakano, Sago & Tanaka 2017]

- Geodesic motion of self-gravitating mass m in *time-symmetric* regular metric $\bar{g}_{\alpha\beta} + h^{\mathrm{R}}_{\alpha\beta}$ derives from canonical Hamiltonian

$$H(x^\mu, p_\mu; \gamma) = \bar{H}(x^\mu, p_\mu) + H_{\text{int}}(x^\mu, p_\mu; \gamma)$$

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- In class of canonical gauges, one can define a *unique* effective Hamiltonian $\mathcal{H}(J) = \bar{H}(J) + \frac{1}{2}\langle H_{\mathrm{int}}\rangle(J)$ yielding a first law valid for *generic* bound orbits:

$$\delta\mathcal{E} = \Omega_\phi\,\delta\mathcal{L} + \Omega_r\,\delta\mathcal{J}_r + \Omega_\theta\,\delta\mathcal{J}_\theta + \langle z\rangle\,\delta m$$

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$$\delta\mathcal{E} = \Omega_\phi \, \delta\mathcal{L} + \Omega_r \, \delta\mathcal{J}_r + \Omega_\theta \, \delta\mathcal{J}_\theta + \langle z \rangle \, \delta m$$

- The actions $\mathcal{J}_\alpha$ and the averaged redshift $\langle z \rangle$, as functions of $(\Omega_r, \Omega_\theta, \Omega_\phi)$, include conservative self-force corrections from the *gauge-invariant* averaged interaction Hamiltonian $\langle H_{\mathrm{int}} \rangle$

Applications of the first laws

- Fix 'ambiguity parameters' in 4PN two-body equations of motion
  [Jaranowski & Schäfer 2012, Damour *et al.* 2014, Bernard *et al.* 2016]
- Inform the 5PN two-body Hamiltonian in a 'tutti-frutti' method
  [Bini, Damour & Geralico 2019, 2020]
- Compute GSF contributions to energy and angular momentum
  [Le Tiec, Barausse & Buonanno 2012]
- Calculate Schwarzschild and Kerr ISCO frequency shifts
  [Le Tiec *et al.* 2012, Akcay *et al.* 2012, Isoyama *et al.* 2014]
- Test cosmic censorship conjecture including GSF effects
  [Colleoni & Barack 2015, Colleoni *et al.* 2015]
- Calibrate EOB potentials in effective Hamiltonian
  [Barausse *et al.* 2012, Akcay & van de Meent 2016, Bini *et al.* 2016]
- Compare particle redshift to black hole surface gravity
  [Zimmerman, Lewis & Pfeiffer 2016, Le Tiec & Grandclément 2018]
- Benchmark for calculations of Schwarzschild IBSO frequency shift and gravitational binding energy [Barack *et al.* 2019, Pound *et al.* 2020]

Summary

- The classical laws of black hole mechanics can be extended to binary systems of compact objects
- First laws of mechanics come in a variety of different forms:
  - Context: exact GR, self-force theory, PN theory
  - Objects: black holes, multipolar point particles
  - Orbits: circular, generic bound
  - Derivation: geometric, Hamiltonian
- Combined with the first law, the redshift $z(\Omega)$ provides crucial information about the binary dynamics:
  - Gravitational binding energy E and angular momentum J
  - ISCO frequency Ω_{ISCO} and IBSO frequency Ω_{IBSO}
  - EOB effective potentials A, $\bar{D}$, Q, ...
  - Horizon surface gravity κ

Prospects

- Small-mass-ratio approximation useful to build templates for IMRIs and even comparable-mass binaries
- Exploit the Hamiltonian first law for a particle in Kerr:
  - Innermost spherical orbits
  - Marginally bound orbits
- Extend Hamiltonian first law for two spinning particles:
  - Non-aligned spins and generic precessing orbits
  - Contribution from quadrupole moments
- Link to unbound orbits and scattering angle via analytic continuation?
- Derive a first law in post-Minkowskian gravity?
- Derive a first law with dissipation?